Stability of Tautological Bundles on Hilbert Schemes of Points on a Surface

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Kurzzusammenfassung

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In dieser Arbeit wird die Theorie der Modulräume von Garben benutzt um in zwei unterschiedlichen Situationen irreduzible symplektische Mannigfaltigkeiten zu studieren. Diese holomorph symplektischen Mannigfaltigkeiten spielen eine wichtige Rolle in der Zerlegung von kompakten komplexen Ricci-flachen Kählermannigfaltigkeiten. Durch die grundlegende Arbeit von Mukai wurde gezeigt, dass der glatte Teil von Modulräumen von Garben auf symplektischen Flächen ebenfalls symplektisch ist. Da die Liste von Beispielen für irreduzible symplektische Mannigfaltigkeiten recht kurz ist, stellt Mukais Resultat eine wichtige Quelle zur Konstruktion und Analyse dieser Mannigfaltigkeiten dar. In dem wir versuchen Mukais Konzept zu verallgemeinern, suchen wir stabile Garben auf höherdimensionalen irreduziblen symplektischen Mannigfaltigkeiten. Allerdings gibt es in der Literatur so gut wie keine Beispiele. Eines der prominentesten Beispiele für irreduzible symplektische Mannigfaltigkeiten sind Hilbertschemata von Punkten auf $K3$ Flächen. Eine große Klasse von expliziten Beispielen von Garben auf Hilbertschemata besteht aus der Klasse der *tautologischen Garben.* Diese werden mit Hilfe einer Fourier−Mukai Transformation aus Garben auf der zugrundeliegenden Fläche konstruiert. In dieser Arbeit zeigen wir die Stabilität von tautologischen Garben kleinen Ranges auf Hilbertschemata von zwei bzw. drei Punkten bezüglich geeignet gewählter Polarisierungen. Diese Resultate werden auch auf den Fall von Hilbertschemata von Punkten auf Abelschen Flächen übertragen. Diese Hilbertschemata spielen eine wichtige Rolle, da sie als abgeschlossene Unterschemata die verallgemeinerten Kummer Varietäten enthalten. Diese wiederum stellen eine weitere wichtige Klasse von Beispielen für irreduzible symplektische Mannigfaltigkeiten dar. Wir zeigen die Stabilität der Einschränkungen von tautologischen Garben kleinen Ranges auf die verallgemeinerten Kummer Varietäten der Dimensionen zwei (die Kummer Fläche) und vier.

Ein mächtiges Werkzeug für die Analyse von irreduziblen symplektischen Mannigfaltigkeiten ist die Betrachtung von Automorphismen endlicher Ordnung. In einer kürzlich enstandenen Zusammenarbeit mit H. Ohashi haben wir ein neues Beispiel einer 19-dimensionalen Familie von nicht-symplektischen Involutionen auf Mannigfaltigkeiten konstruiert, die deformationsäquivalent zu Hilbertschemata von zwei Punkten auf $K3$ Flächen sind: Betrachten wir eine $K3$ Fläche mit Involution und einen Modulraum von Garben auf dieser Fläche. Unter bestimmten technischen Voraussetzungen ist dieser deformationsäquivalent zum Hilbertschema von zwei Punkten und wir erhalten eine induzierte Involution auf diesem Modulraum. Dieser allgemeinen Idee folgend konstruieren wir das neue Beispiel und analysieren den Fixpunktort der Involution. Wir zeigen, dass sich dessen Topologie von der bereits bekannten natürlichen Involution auf Hilbertschemata von zwei Punkten unterscheidet.

Abstract

Keywords: moduli spaces, hyperkähler manifolds, vector bundles

In this thesis the theory of moduli spaces of sheaves is used in two different situations to analyse irreducible symplectic manifolds. These are holomorphic symplectic manifolds which play a central role in the decomposition of compact complex Ricci-flat Kähler manifolds. By the seminal work of Mukai the smooth part of moduli spaces of sheaves on symplectic surfaces is again a symplectic manifold. Since not many examples of irreducible symplectic manifolds are known, this result constitutes a fruitful source to construct and analyse these kinds of manifolds. Trying to generalise Mukai's concept, we look for stable sheaves on higher dimensional irreducible symplectic manifolds. Almost no results have been known in this direction. One of the most important examples of irreducible symplectic manifolds are Hilbert schemes of points on $K3$ surfaces. A big class of explicit examples of sheaves on Hilbert schemes is the class of *tautological sheaves.* These are constructed from sheaves on the underlying $K3$ surfaces by means of a Fourier−Mukai transform. In this thesis we prove the stability of low rank tautological sheaves with respect to well chosen polarisations on the Hilbert schemes of two and three points. We transfer the results to the case of Hilbert schemes of points on abelian surfaces. These Hilbert schemes also play a prominent role since they contain as closed subschemes the generalised Kummer varieties associated with the underlying abelian surface. These varieties constitute another important class of examples of irreducible symplectic manifolds. We prove stability of the restrictions of low rank tautological sheaves to the generalised Kummer varieties of dimension two (the Kummer surface) and four.

A powerful method to analyse irreducible symplectic manifolds is the study of finite order automorphisms. In a recent joint work with H. Ohashi we constructed a new example of a 19-dimensional family of non-symplectic involutions on manifolds deformation equivalent to Hilbert schemes of two points on $K3$ surfaces: We start with a $K3$ surface carrying an involution and consider a moduli space of sheaves on this surface. Under certain technical assumptions the moduli space is deformation equivalent to a Hilbert scheme of points and we obtain an induced involution on this moduli space. Following this general idea, we construct the new example and then analyse the fixed locus of the involution. We prove that its topology is different from the case of the well-known natural involution on Hilbert schemes of two points.

Contents

0 Introduction

During the last thirty years special manifolds named *irreducible holomorphic symplectic manifolds* gained a lot of attention by complex geometers. The importance of these manifolds is due to the fact that they occur as natural building blocks in the decomposition of all compact Ricci-flat Kähler manifolds: The decomposition theorem called *Beauville–Bogomolov decomposition* states that every compact complex Ricci-flat Kähler manifold is — up to a finite étale covering — isomorphic to a product of manifolds where each factor belongs to one of the following three classes: The most classical examples of Ricci-flat manifolds are complex tori. In every complex dimension there exists exactly one deformation type. Secondly we have *strict Calabi–Yau manifolds* which are characterised by being simply connected and do not admit any holomorphic form but in top degree. Starting from dimension three there have been constructed examples of different deformation types in great numbers, e.g. as complete intersections in projective spaces. And finally, there are irreducible holomorphic symplectic manifolds, sometimes called *irreducible symplectic manifolds*. They are simply connected and admit a unique nowhere degenerate holomorphic symplectic two-form.

Examples of irreducible symplectic manifolds have been studied long time before the accurate definition above was available, namely $K3$ surfaces. In the classification of complex surfaces they are distinguished by being the only minimal, regular (i.e. $\mathrm{H}^1(\mathcal{O}) = 0$) surfaces with trivial canonical bundle. They are exactly the two dimensional examples of irreducible symplectic manifolds. By now the theory of $K3$ surfaces is very well developed. There is a well studied theory of moduli spaces of $K3$ surfaces and the powerful instrument of the global Torelli theorem ([PS71]).

Further examples were given in the seminal work of Beauville ([Beau83]): He described two infinite families of higher dimensional irreducible symplectic manifolds: the Hilbert schemes of points on $K3$ surfaces and the generalised Kummer varieties of an abelian surface. In [Muk84] Mukai showed that (the smooth locus of) moduli spaces of sheaves on $K3$ and abelian surfaces carry a natural symplectic structure and it has been shown (cf. [HL97, Sect. 6.2] in the $K3$ and [Yosh01] in the abelian surface case) that they are irreducible symplectic manifolds which are deformation equivalent to Hilbert schemes or generalised Kummer varieties (in the case the moduli spaces are smooth). For a long time these were the only known examples until the surprising discoveries of O'Grady: He considered two families of moduli spaces of sheaves (one on $K3$ surfaces and one on abelian surfaces) which are not smooth and constructed a symplectic desingularisation. Due to the joint effort of several authors (cf. [KLS06], [Zow11]) we know now that the two cases discovered by O'Grady are the only cases which admit a symplectic resolution. Since then no new deformation types of irreducible symplectic manifolds have been discovered: all other approaches yielded only manifolds which are deformation equivalent to one of the known examples. Thus the search for new examples is still one of the big goals in the theory.

Meanwhile much progress has been made in the general theory of irreducible symplectic manifolds. An overview can be found in [Huy99], where the author summarised fundamental results on irreducible symplectic manifolds and formulated important questions for further research. The biggest success of recent years certainly constitutes the proof of an analogue of the global Torelli theorem by Verbitsky (cf. [Mar09] and [Huy10]).

One of the main intentions of this thesis is to propose a new approach for the search for new examples as follows. The result of Mukai (as stated above) shows that when we start with a symplectic surface and consider a moduli space of sheaves on it, we again obtain a symplectic variety. Now applying this principle to higher dimensions, we may formulate the following maxim:

> *Let X be a higher dimensional irreducible symplectic manifold. What can be said about stable sheaves on X and their moduli?*

To date almost no results in this direction are known. In [Schl10] the stability of a small class of certain rank two bundles so-called *tautological bundles* associated with line bundles on the Hilbert scheme of two points on a surface has been proven. More generally *tautological sheaves* are sheaves on Hilbert schemes of points on surfaces which are obtained via a Fourier–Mukai transform from sheaves on the surface. As the kernel for the transform one uses the structure sheaf of the universal subscheme of the Hilbert scheme. These objects have been intensively studied by different authors:

Tikhomirov derived formulas for the top Segre class of tautological sheaves associated with line bundles in [Tik92]. In [Leh99] Lehn studied the action of the algebra spanned by Chern classes of tautological sheaves on the bigraded Hopf algebra $\mathbb{H} := \bigoplus_n \mathrm{H}^*(X^{[n]}, \mathbb{Q})$ and Ellingsrud, Göttsche and Lehn computed the Euler characteristics of tautological sheaves in [EGL01]. Motivated by the study of the so-called *strange duality* Danila ([Dan01]) and Scala ([Scal09a] and [Scal09b]) — both students of Le Potier — computed the spaces of global sections of these objects. In his thesis Krug (cf. [Kru11]) recently gave a complete computation of extension groups of tautological sheaves.

We are able to generalise Schlickewei's result drastically. We analyse the stability of rank two and four tautological sheaves on the Hilbert scheme of two points on a regular or abelian surface. Furthermore, we prove the stability of rank three tautological sheaves on the Hilbert scheme of three points again on regular and abelian surfaces. Finally, we study the restriction of tautological sheaves on Hilbert schemes of abelian surfaces to the associated generalised Kummer varieties of dimension two and four and prove stability in the rank two (rank three resp.) case. Since the two dimensional Kummer variety is nothing but the Kummer surface of the underlying abelian surface, this yields a construction of stable sheaves on Kummer surfaces, a topic of independent interest. A very subtle part in all these constructions is the choice of a suitable polarisation on the manifold one is considering. The results concerning the stability of tautological sheaves on the Hilbert scheme of two points on a regular surface together with the necessary geometric considerations are contained in the preprint [W12].

The deformations of a tautological sheaf can be divided into two classes. The first class consists of deformations which are induced by deformations of the original sheaf on the surface and one can reduce their study to the deformation theory of sheaves on surfaces. This leads to an embedding of the moduli space of sheaves on the surface into the moduli space of tautological sheaves. In many cases tautological sheaves have more deformations. This second class will be called the *additional deformations* and they may lead to singularities of the moduli space of tautological sheaves.

A very fruitful method to study irreducible symplectic manifolds is the theory of finite order automorphisms of these manifolds. In the case of $K3$ surfaces Nikulin ([Nik80a], [Nik80b]) developed the fundamental theory. He used the global Torelli theorem in order to apply his lattice-theoretic results. Since then much progress has been made in the theory of automorphisms of $K3$ surfaces (cf. [Muk88], [Kon98], [GS07], [AST11]). In the higher dimensional case — apart from sporadic examples — automorphisms have been studied systematically in [Beau83b], [BNS11], [OS11], [BNS12], [Cam12] and [Mon12]. An important class of autmorphisms on manifolds of $K3^{[n]}$-type are obtained as follows: Consider an automorphism of finite order on a $K3$ surface. This induces an automorphism on the Hilbert scheme of n points for all n. Boisière called automorphisms on the Hilbert scheme of this kind *natural* (cf. [Boi12]). In [BS12] a characterisation of natural automorphisms was given. More generally, Mongardi called an automorphism on a manifold of $K3^{[n]}$-type *standard* if the pair consisting of the manifold and the automorphism can be deformed to a Hilbert scheme with a natural automorhpism. Beauville gave a rough classification of non-symplectic involutions of manifolds deformation equivalent to the Hilbert scheme of two points on a $K3$ surface in [Beau11]. In a recent prepint ([OW13]) H. Ohashi and myself study the case of 19-dimensional families of such involutions in more detail, give a lattice-theoretic classification and construct a new example.

The construction of this new example follows the following more general concept which should be regarded as the second important maxim of this thesis: Consider a surface together with an automorphism. If the induced action on the cohomology fixes a polarisation and a Mukai vector v, then we end up with an induced automorphism on the moduli space of stable sheaves of Mukai vector v. If the surface is symplectic and the moduli space is smooth, then — under certain technical assumptions — we obtain in this way examples of automorphisms on irreducible symplectic manifolds. A class of symplectic surfaces which have an interesting group of automorphisms consists of Kummer surfaces whence the aforementioned interest in constructions of stable sheaves on Kummer surfaces.

The lattice-theoretic classification of 19-dimensional families of non-symplectic involutions as stated in Theorem 7.3 is due to H. Ohashi. Thus the proof of this statement is omitted in this thesis. The interested reader is referred to [OW13]. All proofs enclosed in Chapter 7 are due to myself.

0.1 Summary of the results

Let (X, H) be a polarised smooth projective surface over the complex numbers. Assume that X is either regular ($h^1(X, \mathcal{O}_X) = 0$) or abelian with Picard rank one (in which case H will be chosen to be a principal polarisation).

Let $X^{[n]}$ be the Hilbert scheme of n points on X. In the product $X \times X^{[n]}$ we have the universal subscheme $\Xi_n := \{(x, \xi) \mid x \in \xi\}$ and there are the projections $q\colon X \times X^{[n]} \to X$ and $p\colon X \times X^{[n]} \to X^{[n]}$. Let $\mathcal{F}$ be a sheaf on X. We define the *tautological sheaf associated with* $\mathcal{F}$ as

$$\mathcal{F}^{[n]} := p_\star(q^\star\mathcal{F} \otimes \mathcal{O}_{\Xi_n}).$$

Theorem (Theorems 4.6, 4.8, 4.13, 5.16, 5.17, 5.18 and Proposition 4.12)**.** *Let* $\mathcal{F}$ *be a* μ_H*-stable sheaf on* X *such that* $\det\mathcal{F} \not\simeq \mathcal{O}_X$*. There is a polarisation on* $X^{[2]}$ *such that if* $\mathcal{F}$ *is of rank one (rank two), then* $\mathcal{F}^{[2]}$ *is a rank two (rank four, respectively)* μ*-stable sheaf on* $X^{[2]}$*. Similarly there is a polarisation on* $X^{[3]}$ *such that if* $\mathcal{F}$ *is of rank one, then* $\mathcal{F}^{[3]}$ *is* μ*-stable of rank three. For arbitrary* n *there is a polarisation on* $X^{[n]}$ *such that* $\mathcal{F}^{[n]}$ *does not contain any destabilising subsheaves of rank one.*

If X is abelian, then inside the Hilbert scheme $X^{[n]}$ there is the generalised Kummer variety $K_n(X)$. Let us denote the embedding by j. On X we have the natural involution ι from the group structure. A sheaf $\mathcal{H}$ is called *symmetric* if $\iota^\star\mathcal{H} \simeq \mathcal{H}$. We have the following results:

Theorem (Theorems 5.27, 5.29 and 5.32)**.** *Let* $\mathcal{F}$ *be a* μ_H*-stable sheaf on* X *such that* $\det\mathcal{F} \not\simeq \mathcal{O}_X$*. There is a polarisation on* $K_3(X)$ *such that if* $\mathcal{F}$ *is of rank one,* $j^\star\mathcal{F}^{[3]}$ *is* μ*-stable of rank three. Furthermore, there is a polarisation on* $K_2(X)$ *(the Kummer surface associated with* X*) such that if* $\det\mathcal{F}$ *is not symmetric and* $\mathcal{F}$ *is of rank one (rank two), the restriction* $j^\star\mathcal{F}^{[2]}$ *is* μ*-stable of rank two (rank four).*

Furthermore, we have the following relation between moduli spaces of sheaves on $K3$ surfaces and moduli spaces of tautological sheaves:

Proposition (Proposition 6.4)**.** *Let* $\mathcal{F}$ *be a stable sheaf on a* $K3$ *surface* X *of Mukai vector* v *such that* $\mathcal{F}^{[2]}$ *is stable (of Mukai vector* $v^{[2]}$*). We have an embedding of moduli spaces* $\mathcal{M}^s(v) \hookrightarrow \mathcal{M}^s(v^{[2]})$.

A key result to construct automorphisms on moduli spaces of sheaves is the following:

Proposition (Proposition 2.35)**.** *Let* (X, H) *be a polarised smooth projective variety and* φ *an automorphism of* X *preserving* H*. Consider a Mukai vector* $v \in \mathrm{H}^*(X,\mathbb{Z})$ *which is invariant under the induced action of* φ*. Then* φ *induces a biregular automorphism* ι *on* $\mathcal{M}^s(v)$*.*

Using this proposition, in a recent joint work with H. Ohashi we proved the following result which is also enclosed in this thesis:

Theorem (Theorems 7.9 and 7.16)**.** *There is a* 19*-dimensional family of manifolds of* $K3^{[2]}$*-type admitting a non-symplectic involution with invariant lattice isomorphic to* U*. Every member of this family is isomorphic to a moduli space of sheaves* $\mathcal{M}(2, H, 0)$ *on a degree two polarised* $K3$ *surface* (X, H) *admitting a double cover to* $\mathbb{P}^2$*. This family is different from the* 19*-dimensional family of natural non-symplectic involutions on the*

Hilbert schemes of two points. The fixed locus of this involution consists of two smooth connected surfaces which are both branched covers of $\mathbb{P}^2$ *of degree six and ten, respectively.*

0.2 Structure of the thesis

This thesis starts with three introductory chapters which recall the most important definitions and facts necessary for the sequel and fix notations. In Chapter 1 we recall the basic facts on irreducible symplectic manifolds, give a list of examples and collect some more detailed geometric aspects of the Hilbert scheme of points on a surface.

The main technical tools we use in this thesis are stable sheaves and their moduli. Thus we summarise the basics on stability of sheaves and their moduli in Chapter 2. We give a more detailed account on sheaves on $K3$ surfaces and add some considerations on induced automorphisms on moduli spaces of sheaves.

The notion of *tautological sheaves* — though they are very natural objects to study — might not be known to all complex geometers working with irreducible symplectic manifolds. Therefore we give their definition and collect the known facts about their cohomology and extension groups in Chapter 3. We also enclose several considerations on polarisations on Hilbert schemes.

In Chapter 4 we start studying the stability of tautological sheaves in the case of regular surfaces. The focus here lies, of course, on $K3$ surfaces. We give a quite detailed proof of the stability results which will provide a model for the proofs in the case of abelian surfaces and Kummer varieties in Chapter 5 where the geometries involved are more difficult to handle.

Deformations and moduli spaces of tautological sheaves are discussed in Chapter 6.

Finally, we discuss several aspects of non-symplectic involutions on manifolds of $K3^{[2]}$-type in Chapter 7. We state a lattice-theoretic classification of 19-dimensional families of these involutions, construct a new example of a non-natural involution and study its fixed locus.

0.3 Notations and conventions

- The base field of all varieties and schemes in this thesis is the field of complex numbers.
- A *lattice* $(L, (\ ,\ ))$ is a free abelian group L together with a non-degenerate symmetric form $(\ ,\ ) \colon L \times L \to \mathbb{Z}$.
- We denote by U the lattice with intersection matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and by E_8 the unique positive-definite, even, unimodular lattice of rank eight.
- For any lattice $(L, (\ ,\ ))$ we write $L(a)$, $a \in \mathbb{Z}$ to denote the lattice with intersection product multiplied by a.
- For the intersection product inside the Chow or cohomology ring of a smooth variety we write either $l.m$ or $l \cdot m$, for classes l and m, or we will just use juxtaposition lm.

- A *polarisation* on a variety X is the choice of a class H inside the ample cone $\operatorname{Amp}(X)$.

- A *sheaf* on a variety X is a coherent $\mathcal{O}_X$-module.

- We write $\cong$ to indicate an isomorphism of abelian groups, vector spaces and varieties. We use $\simeq$ for isomorphisms of sheaves.

- All functors such as pushforward, pullback, local and global homoorphisms and tensor product are not derived unless mentioned otherwise.

- Let $\mathcal{F}$ be a locally free sheaf. A locally free subsheaf $\mathcal{L}$ is called a *subbundle* of $\mathcal{F}$. (In literature this is sometimes only used for locally free subsheaves $\mathcal{L}$ such that the quotient $\mathcal{F}/\mathcal{L}$ is locally free, too.)

- Let $\sigma\colon Y \to Z$ be a morphism of smooth projective varieties. For all sheaves $\mathcal{H}$ on Y and $\mathcal{G}$ on Z there is an adjunction isomorphism of $\mathbb{C}$-vector spaces
$$\operatorname{Hom}_Y(\sigma^\star\mathcal{G}, \mathcal{H}) \cong \operatorname{Hom}_Z(\mathcal{G}, \sigma_\star\mathcal{H})$$
which will be denoted by $\sigma^\star \dashv \sigma_\star$ ($\sigma^\star$ is left adjoint to $\sigma_\star$).

- Let $Y \times Z$ be the product of two varieties Y and Z. Denote the projections to the corresponding factors by π_1 and π_2. For sheaves $\mathcal{H}$ on Y and $\mathcal{G}$ on Z we define
$$\begin{aligned} \mathcal{H} \boxtimes \mathcal{G} &:= \pi_1^\star\mathcal{H} \otimes \pi_2^\star\mathcal{G} \text{ and} \\ \mathcal{H} \boxplus \mathcal{G} &:= \pi_1^\star\mathcal{H} \oplus \pi_2^\star\mathcal{G}. \end{aligned}$$
They are called *exterior tensor product* and *exterior sum* of the sheaves $\mathcal{H}$ and $\mathcal{G}$. In the case $Y = Z$ and $\mathcal{H} = \mathcal{G}$ we also write $\mathcal{H}^{\boxtimes 2}$ and $\mathcal{H}^{\boxplus 2}$. This, of course, generalises to products of more than two varieties.

- Let G be a finite group acting on a smooth projective variety X. Let $f\colon X \to X/G$ be the quotient. If $\mathcal{L}$ is a G-equivariant line bundle on X, the pushforward $f_\star\mathcal{L}$ inherits the G-linearisation. Since G acts trivially on X/G, we can define $\mathcal{L}^G$ to be the sheaf of G-invariant sections of $f_\star\mathcal{L}$, which is a line bundle on X/G. Conversely, the pullback gives a homomorphisms $f^\star\colon \operatorname{Pic}(X/G) \to \operatorname{Pic}^G(X)$ to the group of G-linearised line bundles on X. This map is injective and its image coincides with line bundles $\mathcal{L}$ such that for every $x \in X$ the stabiliser group G_x acts trivially on the fibre $\mathcal{L}_x$ (cf. [KKV89]). Finally, by taking first Chern classes, for every class l in $\operatorname{NS} X$ we can define a class $l^G \in \operatorname{NS}(X/G)$.

- Let Y be a smooth projective variety Y and $\mathcal{E}$ a vector bundle. There is an extension
$$\operatorname{At}(\mathcal{E}) \in \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_Y)$$
called the *Atiyah class* of $\mathcal{E}$. It was introduced by Atiyah in [Ati57] as an obstruction class for the existence of connections on principal bundles. This class satisfies the following properties:

– Denote by $\operatorname{tr}\colon \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_Y) \to \mathrm{H}^1(Y, \Omega_Y)$ the natural trace map. We have

$$\operatorname{tr} \operatorname{At}(\mathcal{E}) = c_1(\mathcal{E}).$$

– For any morphism $f\colon Z \to Y$ there is an induced map $\tilde{f}\colon \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_Y) \to \operatorname{Ext}^1(f^\star\mathcal{E}, f^\star\mathcal{E} \otimes \Omega_Z)$. The Atiyah class satisfies

$$\tilde{f}(\operatorname{At}(\mathcal{E})) = \operatorname{At}(f^\star\mathcal{E}).$$

– If $\mathcal{E}'$ is another vector bundle on Y, we have

$$\operatorname{At}(\mathcal{E} \otimes \mathcal{E}') = \operatorname{At}(\mathcal{E}) \otimes \mathrm{id} + \mathrm{id} \otimes \operatorname{At}(\mathcal{E}').$$

More generally, the Atiyah class can be defined for any object in the derived category by using locally free resolutions. For a discussion of some of the properties of the Atiyah class in a more modern language and in the realm of algebraic geometry the interested reader is referred to [HL97, Sect. 10.1].

- Furthermore, we use the following notation:

$\mathcal{T}_X$	–	tangent sheaf of a variety X
Ω^k_X	–	sheaf of k-th order differentials
ω_X	–	canonical line bundle
ω_f	–	relative canonical sheaf of a morphism f
td_X	–	Todd class of a variety X
td_f	–	Todd class of the relative tangent bundle of a morphism f
b_i	–	i-th Betti number
c_i	–	i-th Chern class
H^*	–	cohomology ring of a variety or collection of all cohomology groups of a sheaf
A_*	–	Chow ring
χ	–	holomorphic Euler characteristic
e	–	topological Euler characteristic
$\mathrm{NS}\,X$	–	Néron−Severi group of a variety X
$\mathrm{Pic}\,X$	–	Picard group of X
$\mathrm{Num}(X)$	–	$\mathrm{Pic}\,X/\sim$, where $\sim$ is numerical equivalence
$\mathrm{Amp}(X)_{\mathbb{Q}}$	–	rational ample cone
$v^{\perp}$	–	orthogonal complement inside a lattice
$(-)^{\mathfrak{S}_n}$	–	invariant part with respect to the $\mathfrak{S}_n$-action
S^nX	–	n-th symmetric product of the variety X
S^nV	–	n-th symmetric product of the vector space V
$V^{\vee}$	–	dual of the vector space V
$\mathcal{F}^{\vee}$	–	dual of the sheaf $\mathcal{F}$ (defined as $\mathcal{H}om(\mathcal{F},\mathcal{O})$)
$h^i(\mathcal{F})$	–	$\dim_{\mathbb{C}} \mathrm{H}^i(\mathcal{F})$

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1 Irreducible Symplectic Manifolds

In this chapter we collect the most important definitions and results about irreducible holomorphic symplectic manifolds. More detailed presentations can be found in [Beau83] and [Huy99].

1.1 Definition and the Beauville–Bogomolov Decomposition

Definition 1.1. Let X be a compact complex Kähler manifold. We call X an *irreducible holomorphic symplectic manifold* if

- X is simply connected and
- $\mathrm{H}^0(X, \Omega_X^2) \cong \mathbb{C}\omega$,

where ω is a nowhere degenerate holomorphic two-form.

Corollary 1.2. *If X is an irreducible holomorphic symplectic manifold of dimension n, we have*

- $2|n$,
- $\omega_X \simeq \mathcal{O}_X$,
- $\mathcal{T}_X \simeq \Omega_X$,
- $h^l(X, \mathcal{O}_X) = h^0(X, \Omega_X^l) = 0$ *for l odd and*
- $\mathrm{H}^0(X, \Omega_X^{2k}) \cong \mathbb{C}\omega^k$ *for $k = 0, \ldots, \frac{n}{2}$.*

Remark 1.3. A compact complex Kähler manifold is irreducible symplectic if and only if it admits a Riemannian metric with holonomy group equal to the symplectic group $Sp(k)$, where $4k$ is the real dimension. Such a manifold is also referred to as *hyperkähler* manifold. A comparison of the two notions can be found in [Beau83].

The importance of irreducible symplectic manifolds is due to the fact that they appear as basic building blocks in the classification of compact complex Ricci flat manifolds. We refer to [Beau83] for a systematic review of results of de Rham, Berger, Bogomolov and Yau which ultimately yield the following theorem.

Theorem 1.4 (Beauville–Bogomolov decomposition)**.** *Let X be a compact complex Ricci-flat Kähler manifold. There exists a finite étale covering $\tilde{X} \to X$ such that*

$$\tilde{X} \cong T \times \Pi_i V_i \times \Pi_j X_j,$$

where T is a complex torus, the V_i are strict Calabi–Yau manifolds and the X_j are irreducible symplectic.

A *strict Calabi–Yau manifold* V is a simply connected compact complex Kähler manifold satisfying $h^0(V, \Omega_V^l) = 0$ for all $0 < l < \dim V$.

An important invariant of any irreducible symplectic manifold is the following:

Theorem 1.5 (Fujiki–Beauville–Bogomolov form)**.** *For every irreducible symplectic manifold X there exists a canonically defined non-degenerate pairing $(\ ,\ )_X$ on $\mathrm{H}^2(X, \mathbb{Z})$ called* Beauville–Bogomolov *or sometimes* Fujiki–Beauville–Bogomolov pairing.

Remark 1.6. The only two dimensional examples of irreducible symplectic manifolds are $K3$ surfaces. Being complex surfaces, they carry a natural intersection pairing on the second cohomology by Poincaré duality. A fundamental fact about the Beauville–Bogomolov pairing is that it coincides with the intersection pairing in the case of $K3$ surfaces.

Remark 1.7. As in the case of $K3$ surfaces there is a well developed theory of moduli spaces for irreducible symplectic manifolds with or withouth marking, there is a period domain and map. In this thesis all this technical machinery is not needed. We refer to [GHS12] for a detailed account and further references.

1.2 Examples

Example 1.8 ($K3$ surfaces)**.** The most basic and also oldest examples of irreducible symplectic manifolds are $K3$ surfaces. They are exactly the two dimensional irreducible symplectic manifolds. In the classification of complex surfaces they are distinguished by the properties $\omega_X \simeq \mathcal{O}_X$ and $b_1 = 0$.

There exist many constructions of $K3$ surfaces. The easiest examples of $K3$ surfaces are complete intersections in projective space. For example, every smooth quartic in $\mathbb{P}^3$ and the intersection of a quadric and a cubic in $\mathbb{P}^4$ yield $K3$ surfaces. Another famous example is the double sextic. Let $C \subset \mathbb{P}^2$ be a sextic curve. Consider the double cover $\pi\colon \tilde{X} \to \mathbb{P}^2$ branched along C. For a smooth sextic C this yields a smooth surface $\tilde{X}$ which turns out to be $K3$. If C has simple singularities, we consider its minimal resolution $X \to \tilde{X}$ and obtain a $K3$. This yields a 19-dimensional (27 parameters of a sextic in $\mathbb{P}^2$, 8 dimensions of $PGL(2)$) family of projective $K3$ surfaces.

For every $K3$ surface there is an isomorphism of lattices

$$\mathrm{H}^2(X, \mathbb{Z}) \cong U^3 \oplus E_8(-1)^2.$$

Thus the second Betti number is 22 and we have a 20-dimensional moduli space of $K3$ surfaces.

Example 1.9 (Hilbert schemes of points on a $K3$ surface)**.** Let $n \geq 2$ be an integer and X a $K3$ surface. Consider its n-fold symmetric product $S^nX = X^n/\mathfrak{S}_n$. It is of dimension $2n$ and singular along the locus consisting of non-reduced subschemes $\xi \subset X$, the locus commonly known as the big diagonal. A resolution of singularities is given by the Hilbert scheme of n points on X denoted $\mathrm{Hilb}^n(X)$ or $X^{[n]}$. The resolution $\rho_n\colon X^{[n]} \to S^nX$ is called *Hilbert–Chow morphism* which is a blowup in codimension two. Certainly the Hilbert scheme deforms with the $K3$ surface but a general deformation of a Hilbert scheme is not the Hilbert scheme of a $K3$. In general we call a manifold which is deformation equivalent to a Hilbert scheme of n points on a $K3$ a *manifold of $K3^{[n]}$-type*. We have $b_2 = 23$ and

$$\mathrm{H}^2(X^{[n]},\mathbb{Z}) \cong \mathrm{H}^2(X,\mathbb{Z}) \oplus \langle -2(n-1)\rangle \cong U^3 \oplus E_8(-1)^2 \oplus \langle -2(n-1)\rangle,$$

where a generator of the last summand corresponds to half the class of the exceptional divisor of ρ_n. The moduli space of manifolds of $K3^{[n]}$-type is 21-dimensional.

Example 1.10 (Kummer surfaces)**.** Let us introduce the famous Kummer construction of $K3$ surfaces which also admits a higher dimensional analogue as will be discussed in the next example. Let A be an abelian surface. The natural involution on A given by the inverse $\iota\colon a \mapsto -a$ has 16 fixed points (which are exactly the 16 two-torsion points). Thus if we blow up the 16 A_1-singularities in A/ι, we obtain a smooth surface which can easily be shown to be $K3$. It is the so-called *Kummer surface* $\mathrm{Km}A$ *associated with* A.

An alternative construction of $\mathrm{Km}A$ goes as follows: Let $b\colon \tilde{A} \to A$ denote the simultaneous blowup of all fixed points of the involution ι on A and denote by $E_1,\ldots,E_{16}$ the exceptional divisors. On $\tilde{A}$ we still have an involution which fixes the E_l pointwise. We consider the quotient $\tau\colon \tilde{A} \to \mathrm{Km}A$ which is a degree two covering onto the associated Kummer surface. By [BHPV04, VIII Prop. 5.1] we have a monomorphism

$$\alpha = \tau_! b^\star\colon\ \mathrm{H}^2(A,\mathbb{Z}) \to \mathrm{H}^2(\mathrm{Km}A,\mathbb{Z})$$

satisfying

$$\alpha(x)\alpha(y) = 2xy \text{ for all } x,y \in \mathrm{H}^2(A,\mathbb{Z}).$$

We have an inclusion of finite index

$$\alpha(\mathrm{NS}\,A) \oplus \bigoplus_{l=1}^{16} \mathbb{Z}N_l \subset \mathrm{NS}(\mathrm{Km}A),$$

where $N_l = \tau(E_l)$. It is well known that $E_l^2 = -1$ and $N_l^2 = -2$. Furthermore, the class $\sum_l N_l$ is 2-divisible and we have $\tau^\star(\frac{1}{2}\sum_l N_l) = \sum_l E_l$ and $\tau^\star N_l = 2E_l$.

Finally, we have

$$\mathrm{NS}\,\tilde{A} \cong b^\star\,\mathrm{NS}\,A \oplus \bigoplus_{l=1}^{16} \mathbb{Z}E_l \quad \text{and} \quad \mathrm{Pic}^0\tilde{A} \cong b^\star\,\mathrm{Pic}^0 A. \tag{1}$$

Example 1.11 (Generalised Kummer varieties)**.** Transfering the upper construction of Hilbert schemes to the case of abelian surfaces also yields Ricci flat manifolds. But they are not simply connected and contain additional factors in the Beauville–Bogomolov decomposition. To get rid of these factors we consider (for an abelian surface A) the following composition

$$m_n\colon A^{[n]} \xrightarrow{\rho_n} S^n A \xrightarrow{\tilde{s}_n} A,$$

where $\tilde{s}_n$ is the summation map. All fibres of m_n are isomorphic and we define

$$K_n(A) := m_n^{-1}(0)$$

and call it *generalised Kummer variety.* It is a $(2n-2)$-dimensional irreducible symplectic manifold (cf. [Beau83]). In the case $n = 2$ this just gives back the Kummer surface $\mathrm{Km}A$. For all $n > 2$ we have $b_2 = 7$ and

$$\mathrm{H}^2(K_n(A), \mathbb{Z}) \cong \mathrm{H}^2(A, \mathbb{Z}) \oplus \langle -2n \rangle \cong U^3 \oplus \langle -2n \rangle$$

and in this case a generator of the last summand is half the class of the restriction of the exceptional divisor of the Hilbert−Chow morphism to the Kummer variety.

Note that some authors (e.g. [Beau83]) use the notation K_n for the generalised Kummer variety of dimension $2n$. We will use the notation introduced above which is also used in [Huy99].

Example 1.12 (Moduli spaces of sheaves)**.** A subscheme $\xi \subset X$ of a $K3$ surface is uniquely determined by its ideal sheaf $\mathcal{I}_\xi$. Thus the Hilbert scheme $X^{[n]}$ can also be seen as a moduli space of ideal sheaves. More generally, Mukai ([Muk84]) has proven that on (the smooth locus of) moduli spaces of sheaves on $K3$ and abelian surfaces there is a symplectic structure. This construction leads to irreducible symplectic manifolds deformation equivalent to Hilbert schemes and generalised Kummer varieties described above. More details can be found in Section 2.3.

Example 1.13 (O'Grady's examples)**.** For a long time the above mentioned examples were the only known examples. In [O'G99] and [O'G03] O'Grady considered singular moduli spaces of sheaves on $K3$ and abelian surfaces and constructed symplectic desingularisations which led to two new deformation types of irreducible symplectic manifolds. Finally, Kaledin−Lehn−Sorger ([KLS06]) and Zowislok ([Zow11]) proved that O'Grady's examples are the only cases where a singular moduli space of sheaves on a $K3$ or abelian surface admits a symplectic desingularisation.

1.3 On the Geometry of Hilb^2

In this section we want to collect all important geometric properties of the Hilbert scheme of two points on a smooth projective surface X.

Consider the following basic blowup and projections diagram:

$$
\begin{tikzcd}
D \arrow[rr, "i"] \arrow[d, "\sigma_D"] & & \widetilde{X \times X} \arrow[d, "\sigma"] \arrow[ddl, "r_1"'] \arrow[ddr, "r_2"] \arrow[drr, "\psi"] & & & X \times X^{[2]} \arrow[dl, "p"] \arrow[dr, "q"] & \\
X \arrow[rr, "\Delta"] & & X \times X \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[drr] & & X^{[2]} \arrow[d, "\rho"] & & X \\
& X & & X & S^2X & &
\end{tikzcd}
.
$$

Here Δ is the diagonal embedding, σ is the blowup morphism (we are blowing up the diagonal), $D \simeq \mathbb{P}(\mathcal{N}_{X|X\times X}) \simeq \mathbb{P}(\mathcal{T}_X)$ denotes the exceptional divisor together with the projection σ_D and the inclusion i. We denote the dual of the tautological line bundle of D by $\mathcal{O}_D(1)$. It is well known that $\mathcal{N}_{D|\widetilde{X\times X}} \cong \mathcal{O}_D(-1)$ (see, for example, Theorem II 8.24 in [Har77]). Furthermore, π_1, π_2, p and q denote the natural projections onto the particular factors and r_1 and r_2 are the compositions of π_1 and π_2 with σ. Last but not least we have the flat degree two covering ψ and the Hilbert$-$Chow morphism ρ.

Remark 1.14. Note that for $n = 2$ the universal subscheme $\Xi_2 := \{(x, \xi) \mid x \in \xi\} \subset X \times X^{[2]}$ is isomorphic to $\widetilde{X \times X}$ and the restriction of the two projections p and q above correspond to ψ and r_1.

Next, let us summarise the most important facts about the Chow rings of the varieties involved in the upper diagram. We will follow very closely [Ful84], Sections 6.7 and 15.4, especially Lemma 15.4. On $D = \mathbb{P}(\mathcal{T}_X)$ we have the short exact sequence: $0 \to \mathcal{O}_D(-1) \to \sigma_D^\star \mathcal{N}_{X|X\times X} \to \mathcal{Q} \to 0$, where $\mathcal{Q}$ is the universal quotient line bundle. We have $\mathcal{N}_{X|X\times X} \simeq \mathcal{T}_X$ and — by comparing Chern classes — can therefore see that $\mathcal{Q} \simeq \mathcal{O}_D(1) \otimes \sigma_D^\star \omega_X^\vee$:

$$0 \to \mathcal{O}_D(-1) \to \sigma_D^\star \mathcal{T}_X \to \mathcal{O}_D(1) \otimes \sigma_D^\star \omega_X^\vee \to 0.$$

Let ξ denote the first Chern class of $\mathcal{O}_D(1)$. By Remark 3.2.4 and Theorem 3.3 in [Ful84] we have

$$A_*(D) \cong A_*(X)[\xi]/(\xi^2 + c_1(\mathcal{T}_X)\xi + c_2(\mathcal{T}_X)).$$

Proposition 6.7e) in [Ful84] describes the structure of $A_*(\widetilde{X \times X})$. We gather the most important identities in this ring in the following lemma. Note that since σ is not flat we use the refined Gysin map $\sigma^\star$ as defined in [Ful84, Sect. 6].

Lemma 1.15. *Let $\alpha, \beta, \gamma \in A_*(X)$, and $\lambda \in A_*(D)$. We have the following identities in*

$A_*(\widetilde{X \times X})$ (*or* $A_*(D)$ *for* b)):

$$\begin{array}{lrcl} \text{a)} & i_\star(\xi \cdot \sigma_D^\star(\alpha)) & = & \sigma^\star\Delta_\star(\alpha) + i_\star\sigma_D^\star(\alpha \cdot \omega_X), \\ \text{b)} & i^\star i_\star\lambda & = & -\xi \cdot \lambda, \\ \text{c)} & i_\star\sigma_D^\star\alpha \cdot \sigma^\star(\beta \otimes \gamma) & = & i_\star\sigma_D^\star(\alpha \cdot \beta \cdot \gamma), \\ \text{d)} & i_\star\sigma_D^\star(\alpha) \cdot i_\star\sigma_D^\star(\beta) & = & -\sigma^\star\Delta_\star(\alpha \cdot \beta) - i_\star\sigma_D^\star(\alpha \cdot \beta \cdot \omega_X), \end{array}$$

where we write $\beta \otimes \gamma$ *for* $\pi_1^\star\beta \cdot \pi_2^\star\gamma$.

Proof. a) This follows from the so-called *key formula* in Prop. 6.7.a) in [Ful84]. Note that in our situation the excess normal bundle is just the universal quotient bundle denoted by $\mathcal{Q}$ above. We have $c_1(\mathcal{Q}) = c_1(\mathcal{O}_D(1) \otimes \sigma_D^\star\omega_X^\vee) = \xi - \sigma_D^\star\omega_X$.
b) This is the self-intersection formula Cor 6.3 in [Ful84]:

$$i^\star i_\star\lambda = c_1(\mathcal{N}_{D|\widetilde{X \times X}}) \cdot \lambda = c_1(\mathcal{O}_D(-1)) \cdot \lambda = -\xi \cdot \lambda.$$

c) We have $\alpha \cdot \beta \cdot \gamma = \alpha \cdot \Delta^\star(\beta \otimes \gamma)$. Applying $\sigma_D^\star$, we get

$$\begin{array}{rcccl} \sigma_D^\star(\alpha \cdot \beta \cdot \gamma) & = & \sigma_D^\star(\alpha \cdot \Delta^\star(\beta \otimes \gamma)) & = & \sigma_D^\star\alpha \cdot \sigma_D^\star\Delta^\star(\beta \otimes \gamma) \\ & & & = & \sigma_D^\star\alpha \cdot i^\star\sigma^\star(\beta \otimes \gamma). \end{array}$$

Now we apply $i_\star$ and use the projection formula.
d) We use the projection formula and then b) to find

$$i_\star\sigma_D^\star(\alpha) \cdot i_\star\sigma_D^\star(\beta) = i_\star(i^\star i_\star\sigma_D^\star(\alpha) \cdot \sigma_D^\star(\beta)) = -i_\star(\xi \cdot \sigma_D^\star(\alpha) \cdot \sigma_D^\star(\beta)) = -i_\star(\xi \cdot \sigma_D^\star(\alpha \cdot \beta)).$$

Now we apply a) and we are done. □

Corollary 1.16. *We write* D *for the class* $i_\star[D] \in A_3(\widetilde{X \times X})$ *and denote with* Δ *the cohomology class of the diagonal in* $X \times X$. *We have*

$$\begin{array}{lrcl} \text{a)} & i^\star D & = & -\xi, \\ \text{b)} & D^2 & = & -i_\star\xi \\ & & = & -\sigma^\star\Delta - i_\star\sigma_D^\star(\omega_X) \text{ and} \\ \text{c)} & (\sigma^\star\Delta)^2 & = & \sigma^\star\Delta_\star(c_2(\mathcal{T}_X)). \end{array}$$

Proof. a) Apply b) of Lemma 1.15 to $\lambda = [D]$.
b) We use a) and for the second equality we apply a) of the Lemma to $\alpha = [X]$ to get

$$D^2 = i_\star i^\star D = i_\star(-\xi) = -\sigma^\star\Delta - i_\star\sigma_D^\star(\omega_X).$$

c) Very similarly to the proof of b) in the lemma we use the self-intersection formula:

$$(\sigma^\star\Delta)^2 = \sigma^\star(\Delta^2) = \sigma^\star\Delta_\star\Delta^\star\Delta = \sigma^\star\Delta_\star\Delta^\star\Delta_\star[X] = \sigma^\star\Delta_\star(c_2(\mathcal{N}_{X|X\times X})) = \sigma^\star\Delta_\star(c_2(\mathcal{T}_X)).$$

□

We will continue by determining the canonical line bundles of D and $\widetilde{X\times X}$. On $\widetilde{X\times X}$ we have a short exact sequence:

$$0 \to \mathcal{T}_{\widetilde{X\times X}} \to \sigma^\star\mathcal{T}_{X\times X} \to i_\star(\mathcal{O}_D(1)\otimes\sigma_D^\star\omega_X^\vee) \to 0.$$

We immediately derive $\omega_{\widetilde{X\times X}} \simeq \sigma^\star(\omega_X^{\boxtimes 2})\otimes\mathcal{O}(D)$.

Next, on D we have the exact sequence:

$$0 \to \mathcal{T}_D \to i^\star\mathcal{T}_{\widetilde{X\times X}} \to \mathcal{O}_D(-1) \to 0.$$

Again, we deduce $\omega_D \simeq (\sigma_D^\star\omega_X^\vee)^{\otimes 2}\otimes\mathcal{O}_D(-2)$.

Let us finish this section with some considerations concerning the Picard and Néron−Severi groups of the varieties we are looking at. We have

$$\operatorname{Pic}^0(\widetilde{X\times X}) \cong \sigma^\star\operatorname{Pic}^0(X\times X) \cong \sigma^\star(\operatorname{Pic}^0 X)^{\boxplus 2}$$

and

$$\operatorname{Pic}^0 X^{[2]} \cong \operatorname{Pic}^0 X.$$

Furthermore, we have primitive embeddings

$$(\operatorname{NS} X)^{\boxplus 2} \hookrightarrow \operatorname{NS}(X\times X) \quad\text{and}\quad (\operatorname{NS} X)^{\boxplus 2}\oplus\mathbb{Z}D \hookrightarrow \operatorname{NS}(\widetilde{X\times X}). \tag{2}$$

For a class $l\in\operatorname{NS} X$ we set

$$l_{X^{[2]}} := (\sigma^\star l^{\boxplus 2})^{\mathfrak{S}_2} = \rho^\star((l^{\boxplus 2})^{\mathfrak{S}_2}) \in \operatorname{NS} X^{[2]}.$$

This gives a primitive embedding

$$\operatorname{NS} X\oplus\mathbb{Z}\delta \hookrightarrow \operatorname{NS} X^{[2]} \tag{3}$$

where δ is a class such that 2δ is the class of the exceptional divisor of the blowup morphism $\rho\colon X^{[2]}\to S^2X$. Note that with this notation we have

$$\psi^\star(l_{X^{[2]}}+a\delta) = r_1^\star l + r_2^\star l + aD \quad \text{on } \widetilde{X\times X}.$$

Similarly we have a primitive embedding

$$(-)_{X^{[2]}}\colon \operatorname{Pic} X \hookrightarrow \operatorname{Pic} X^{[2]},\ \mathcal{L}\mapsto\mathcal{L}_{X^{[2]}},$$

where $\mathcal{L}_{X^{[2]}}$ is defined in exactly the same way as $l_{X^{[2]}}$ above.

The line bundle corresponding to the covering $\psi\colon \widetilde{X \times X} \to X^{[2]}$ is denoted by $\mathcal{O}(\delta)$ since its first Chern class is δ even though it is not effective.

Now assume that X is regular, i.e. $h^1(X, \mathcal{O}_X) = 0$. In this case the embeddings (2) and (3) are isomorphisms. Here we apply Exercise III 12.6b) in [Har77]. Thus we can write every element of $\mathrm{NS}(\widetilde{X \times X})$ as $l_1 \otimes 1 + 1 \otimes l_2 + aD$ for some $l_1, l_2 \in \mathrm{NS}\, X$ and $a \in \mathbb{Z}$ and every element of $\mathrm{NS}\, X^{[2]}$ as $l_{X^{[2]}} + a\delta$ for some $l \in \mathrm{NS}\, X$ and $a \in \mathbb{Z}$.

Finally, let us also consider the case that our surface is an abelian surface A. As before we have $(\mathrm{NS}\, A)^{\boxplus 2} \subset \mathrm{NS}(A \times A)$ but in this case there are additional classes. In general it is very difficult to compute $\mathrm{NS}(A \times A)$ explicitly (cf. Chapter 5). Here let us note only the following: We have the group law $s\colon A \times A \to A$ which factors through $\tilde{s}\colon S^2 A \to A$ and we denote the composition $\tilde{s} \circ \rho$ by m:

$$
\begin{tikzcd}
A^{[2]} \arrow[r, "\rho"] \arrow[dr, "m"'] & S^2 A \arrow[d, "\tilde{s}"] & A \times A \arrow[l] \arrow[dl, "s"] \\
& A &
\end{tikzcd}
\qquad . \tag{4}
$$

Thus we obtain a natural map $s^\star\colon\ \mathrm{NS}\, A \to \mathrm{NS}(A \times A)$. For any class $l \in \mathrm{NS}\, A$ we define

$$l_M := s^\star l - l^{\boxplus 2}.$$

Lemma 1.17. *The class l_M is linearly independent of the summand* $(\mathrm{NS}\, A)^{\boxplus 2}$.

Proof. Certainly l_M is $\mathfrak{S}_2$-invariant. Assume we can write

$$s^\star l - l^{\boxplus 2} = l_M = f^{\boxplus 2}$$

with $f \in \mathrm{NS}\, A$. We fix a point $x \in A$ and a curve C on A. We intersect the above equation with the curve $C_x := \{x\} \times C$. We have $s^\star l \cdot C_x = l.C = (1 \otimes l) \cdot C_x$ and $(l \otimes 1) \cdot C_x = 0$. Thus the left hand side vanishes and on the right hand side we are left with $f.C$. Since C was arbitrary, we must have $f = 0$. $\square$

The index M stands for *Mumford class.* Therefore, we have an embedding

$$(-)_m\colon\ \mathrm{NS}\, A \hookrightarrow \mathrm{NS}\, A^{[2]}, \quad l \mapsto l_m := m^\star l - l_{A^{[2]}}. \tag{5}$$

In Section 5.1 we will consider the case that A is a principally polarised abelian surfaces in more detail. Under some technical assumptions we will show that, in fact, the embedding (5) is primitive and no other additional classes may occur.

1.4 On the Geometry of Hilb^n

If $n > 2$, the geometry of the Hilbert scheme of n points on a smooth projective surface X is much more delicate. An important fact is that the Hilbert−Chow morphism $\rho_n\colon X^{[n]} \to S^n X$ is no longer a global blowup morphism. But following [Beau83], we see that outside codimension two subschemes ρ_n is the blowup of the big diagonal. This is important especially if we want to determine the Picard group of the Hilbert scheme. Nevertheless, the geometry of Hilbert schemes has been intensively studied. We will summarise the important results in this section and — as far as possible — try to imitate the notation of the preceding section.

Following [EGL01, Section 1], we consider the following diagram:

$$\begin{tikzcd}
 & X^{[n-1,n]} \arrow[rr, "w_n"'] \arrow[d, "\sigma_n"] \arrow[drr, "\psi_n"] & & \Xi_n \arrow[r, "q_n"'] \arrow[d, "p_n"'] & X & \\
 & \Xi_{n-1} \subset X \times X^{[n-1]} \arrow[dl, "p_{n-1}"] \arrow[dr, "q_{n-1}"] & & X^{[n]} \arrow[dr, "\rho_n"'] & & X^n \arrow[dl] \\
X^{[n-1]} & & X & & S^n X. &
\end{tikzcd}$$

Here we denote by

$$\Xi_n := \{(x, \xi) \mid x \in \xi\} \subset X \times X^{[n]}$$

the universal subscheme and by

$$X^{[n-1,n]} := \{(\xi', \xi) \mid \xi' \subset \xi\} \subset X^{[n-1]} \times X^{[n]}$$

the so-called *nested Hilbert scheme.*

We have the flat degree n covering $p_n\colon \Xi_n \to X^{[n]}$ which is, in fact, the restriction of the second projection $X \times X^{[n]} \to X^{[n]}$. Furthermore, $X^{[n-1,n]}$ is ismorphic to the blowup of $X \times X^{[n-1]}$ along the universal subscheme Ξ_{n-1}. Denote this blowup morphism by σ_n and the projections from $X \times X^{[n-1]}$ to $X^{[n-1]}$ and X by p_{n-1} and q_{n-1}, respectively. By [ES98, Prop. 2.1] the second projection $\psi_n\colon X^{[n-1,n]} \to X^{[n]}$ factors through Ξ_n and from [Hai01, Prop. 3.5.3] it follows that w_n is an isomorphism outside codimension four subschemes. Thus the morphism ψ_n is flat outside codimension four. Finally, we have $q_{n-1} \circ \sigma_n = q_n \circ w_n$

In analogy to the $n = 2$ case we have

$$\mathrm{Pic}^0 X^{[n]} \cong \mathrm{Pic}^0 X$$

and embeddings

$$(-)_{X^{[n]}}\colon\ \mathrm{NS}\, X \hookrightarrow \mathrm{NS}\, X^{[n]},\ l \mapsto l_{X^{[n]}} := \rho_n^\star (l^{\boxplus n})^{\mathfrak{S}_n} \quad \text{and} \quad (-)_{X^{[n]}}\colon\ \mathrm{Pic}\, X \hookrightarrow \mathrm{Pic}\, X^{[n]}.$$

Furthermore, there is a class $\delta_n \in \mathrm{NS}\, X^{[n]}$, such that $2\delta_n$ is the class of the divisor consisting of all non-reduced subschemes $\xi \subset X$. There is a line bundle $\mathcal{O}(\delta_n)$ with first Chern class δ_n such that its pullback $p_n^\star \mathcal{O}(\delta_n)$ is the relative canonical sheaf of p_n.

Lemma 1.18. *Let D_n be the exceptional divisor of σ_n. We have*

$$\psi_n^\star \mathcal{O}(\delta_n) \simeq \mathcal{O}(D_n) \otimes \sigma_n^\star p_{n-1}^\star \mathcal{O}(\delta_{n-1}).$$

Proof. This is a consequence of the geometric considerations in Section 2 of [EGL01]. We introduce the following notation: Set

$$\begin{aligned} \psi_X &:= \psi_n \times \mathrm{id}_X \\ \phi_n &:= p_{n-1} \circ \sigma_n \\ \phi_X &:= \phi_n \times \mathrm{id}_X \\ j &:= \mathrm{id}_{X^{[n-1,n]}} \times (q_{n-1} \circ \sigma_n) \end{aligned}$$

These maps fit into the following diagram, where π is the first projection:

$$\begin{tikzcd} X^{[n-1,n]} \arrow[r, "j"] & X^{[n-1,n]} \times X \arrow[r, "\pi"] \arrow[dl, "\psi_X"] \arrow[dr, "\phi_X"] & X^{[n-1,n]} \\ X^{[n]} \times X & & X^{[n-1]} \times X \end{tikzcd}.$$

Using this notation, we can write sequence (6) from [EGL01] as follows:

$$0 \to j_\star \mathcal{O}(D_n) \to \psi_X^\star \mathcal{O}_{\Xi_n} \to \phi_X^\star \mathcal{O}_{\Xi_{n-1}} \to 0.$$

To this sequence we want to apply $\pi_\star$. First note that $\pi \circ j = \mathrm{id}_{X^{[n-1,n]}}$. Furthermore, we have a commutative diagram

$$\begin{tikzcd} X^{[n-1,n]} \times X \arrow[r, "\pi"] \arrow[d, "\psi_X"] & X^{[n-1,n]} \arrow[d, "\psi_n"] \\ X^{[n]} \times X \arrow[r, "p_n"] & X^{[n]}, \end{tikzcd}$$

where p_n is the first projection as usual. Since the projections are flat we have

$$\pi_\star \circ \psi_X^\star \simeq \psi_n^\star \circ p_{n\star}.$$

Define

$$\mathcal{O}_X^{[n]} := p_{n\star} \mathcal{O}_{\Xi_n}.$$

This is a rank n vector bundle on $X^{[n]}$ with determinant $\mathcal{O}(\delta_n)$. Thus we find $\pi_\star \psi_X^\star \mathcal{O}_{\Xi_n} \simeq \psi_n^\star \mathcal{O}_X^{[n]}$ and similarly $\pi_\star \phi_X^\star \mathcal{O}_{\Xi_n} \simeq \phi_n^\star \mathcal{O}_X^{[n-1]}$. Altogether we see that we have an exact sequence on $X^{[n-1,n]}$:

$$0 \to \mathcal{O}(D_n) \to \psi_n^\star \mathcal{O}_X^{[n]} \to \phi_n^\star \mathcal{O}_X^{[n-1]} \to 0.$$

Hence taking determinants yields the lemma. $\square$

Corollary 1.19. *We have*

$$\psi_n^\star \delta_n = [D_n] + \sigma_n^\star p_{n-1}^\star \delta_{n-1}.$$

Next, there is a recursive formula for classes in $\mathrm{NS}\, X^{[n]}$ coming from X:

Lemma 1.20. *For every class $l \in \mathrm{NS}\, X$ we have*

$$\psi_n^\star l_{X^{[n]}} = \sigma_n^\star (p_{n-1}^\star l_{X^{[n-1]}} + q_{n-1}^\star l).$$

Proof. Denote the natural degree n covering $S^{n-1} \times X \to S^n X$ by f_n. We have

$$\begin{aligned} \psi_n^\star l_{X^{[n]}} &= \psi_n^\star \rho_n^\star (l^{\boxplus n})^{\mathfrak{S}_n} && = \sigma_n^\star (\rho_{n-1} \times \mathrm{id}_X)^\star f_n^\star (l^{\boxplus n})^{\mathfrak{S}_n} \\ &= \sigma_n^\star (\rho_{n-1} \times \mathrm{id}_X)^\star ((l^{\boxplus n-1})^{\mathfrak{S}_{n-1}} \boxplus l) && = \sigma_n^\star (\rho_{n-1}^\star (l^{\boxplus n-1})^{\mathfrak{S}_{n-1}} \boxplus l) \\ &= \sigma_n^\star (p_{n-1}^\star l_{X^{[n-1]}} + q_{n-1}^\star l). \end{aligned}$$

□

Remark 1.21. We leave it to the reader to formulate and prove the analogous result to the lemma above for line bundles instead of cohomology classes.

As in the $n = 2$ case, if X is regular, we have

$$\mathrm{NS}\, X^{[n]} \cong \mathrm{NS}\, X \oplus \mathbb{Z}\delta_n$$

and if $X = A$ is an abelian surface, we have embeddings

$$\begin{aligned} (-)_{M_n} \colon \mathrm{NS}\, A \hookrightarrow \mathrm{NS}\, A^n, \quad l \mapsto s_n^\star l - l^{\boxplus n} \quad \text{and} \\ (-)_{m_n} \colon \mathrm{NS}\, A \hookrightarrow \mathrm{NS}\, A^{[n]}, \quad l \mapsto m_n^\star l - l_{X^{[n]}}, \end{aligned} \tag{6}$$

where we generalise diagram (4) at the end of Section 1.3 as follows:

$$\begin{tikzcd} A^{[n]} \arrow[r, "\rho_n"] \arrow[dr, "m_n"'] & S^n A \arrow[d, "\tilde{s}_n"] & A^n \arrow[l] \arrow[dl, "s_n"] \\ & A & \end{tikzcd}.$$

2 Moduli Spaces of Sheaves

This chapter serves as an introduction to the theory of stable sheaves and their moduli. The standard reference is [HL97].

2.1 Stable Sheaves

Throughout this section we fix a polarised smooth projective variety (X, H) of dimension n.

Definition 2.1. A coherent sheaf $\mathcal{F}$ of dimension $n = \dim X$ is called *torsion-free* if $\mathcal{F}$ does not contain any subsheaf of smaller dimension.

Definition 2.2. We define the *H-slope* of a sheaf $\mathcal{F}$ on X as

$$\mu_H(\mathcal{F}) := \frac{c_1(\mathcal{F}).H^{n-1}}{\operatorname{rk} \mathcal{F}} \in \mathbb{Q}.$$

Definition 2.3. A sheaf $\mathcal{F}$ on X is called *μ_H-semistable* if $\mathcal{F}$ is torsion-free and for every non-trivial subsheaf $\mathcal{E} \subsetneq \mathcal{F}$ we have

$$\mu_H(\mathcal{E}) \leq \mu_H(\mathcal{F}).$$

Furthermore, $\mathcal{F}$ is called *μ_H-stable* if for all $\mathcal{E}$ the inequality above is strict. Finally, we call a subsheaf $\mathcal{E}$ *destabilising* if $\mu_H(\mathcal{E}) \geq \mu_H(\mathcal{F})$.

Example 2.4. Every torsion-free rank one sheaf is automatically stable. A sheaf $\mathcal{F}$ is stable if and only if $\mathcal{F} \otimes \mathcal{L}$ is stable for any line bundle $\mathcal{L}$. A simple method to construct stable sheaves of higher rank is using extensions. For example, let $\mathcal{E}$ and $\mathcal{G}$ be torsion-free rank one sheaves such that $\mu(\mathcal{G}) - \mu(\mathcal{E}) = 1$, then any non-trivial extension (if it exists)

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

is a stable rank two sheaf (cf. Lemma 7.6).

Lemma 2.5. *For every torsion-free sheaf $\mathcal{F}$ there exists a unique maximal μ_H-semistable subsheaf $\mathcal{E}$ of maximal H-slope, i.e. for all subsheaves $\mathcal{G} \subset \mathcal{F}$ we have $\mu_H(\mathcal{G}) \leq \mu_H(\mathcal{E})$ and $\mathcal{G} \subseteq \mathcal{E}$ in case of equality.*

Proof. We can literally translate the proof of [HL97, Lem. 1.3.5] to the case of slope stability. □

Definition 2.6. The subsheaf $\mathcal{E}$ above is called *maximal destabilising subsheaf.*

Definition 2.7. A subsheaf $\mathcal{E} \subset \mathcal{F}$ is called *saturated* if the quotient $\mathcal{F}/\mathcal{E}$ is torsion-free.

Definition 2.8. A torsion-free sheaf $\mathcal{F}$ is called *reflexive* if the embedding into its so-called *reflexive hull* $\mathcal{F}^{\vee\vee}$ is an isomorphism.

Remark 2.9. A reflexive sheaf is locally free outside a subscheme of codimension at least three. Thus on curves and surfaces every reflexive sheaf is automatically a vector bundle.

Lemma 2.10. *The maximal destabilising subsheaf of a reflexive sheaf is reflexive and saturated.*

Proof. Let $\mathcal{E} \subset \mathcal{F}$ be the maximal destabilising subsheaf with $\mathcal{F}$ reflexive. The reflexive hull $\mathcal{E}^{\vee\vee}$ is a subsheaf of $\mathcal{F}$ of the same rank as $\mathcal{E}$. Thus we have $\mu_H(E^{\vee\vee}) \geq \mu_H(\mathcal{E})$ which yields $\mathcal{E}^{\vee\vee} \simeq \mathcal{E}$. Furthermore, for every subsheaf $\mathcal{E}$ there is the notion of the saturation of $\mathcal{E}$. A similar reasoning as in the case of the reflexive hull also yields the second statement of the lemma. $\square$

There is another notion of stability, usually referred to as *Gieseker stability* or *Gieseker–Maruyama stability.* The definition is similar to the one of μ-stability, only the slope is replaced by the Hilbert polynomial:

Definition 2.11. We define the *reduced Hilbert polynomial of* $\mathcal{F}$ (with respect to H) as

$$p_H(\mathcal{F})(l) := \frac{\chi(\mathcal{F} \otimes \mathcal{O}(lH))}{\operatorname{rk} \mathcal{F}} \in \mathbb{Q}.$$

On the set of rational polynomials we introduce the lexicographic ordering and denote it simply by $\leq$.

Definition 2.12. A sheaf $\mathcal{F}$ on X is called *Gieseker semistable* if for non-trivial subsheaves $\mathcal{E} \subsetneq \mathcal{F}$ we have

$$p_H(\mathcal{E}) \leq p_H(\mathcal{F}).$$

Furthermore, $\mathcal{F}$ is called *Gieseker stable* if for every $\mathcal{E}$ the inequality above is strict.

Remark 2.13. To compute and compare Hilbert polynomials of sheaves can be very intricate especially if the dimension of the underlying variety gets big. The computation of slopes is, usually, much simpler. Thus it is much easier to decide if a given sheaf is slope stable or not than in the case of Gieseker stability.

Example 2.14. If X is a curve, then Gieseker stability and slope stability coincide and are both independent of the polarisation H. The slope of a sheaf $\mathcal{F}$ is then identified with $\deg \mathcal{F} / \operatorname{rk} \mathcal{F}$.

As a direct consequence of the definitions we have the following relation between slope stability and Gieseker stability:

Lemma 2.15. *We have the following implications:*

$\mathcal{F}$ is μ-stable $\Rightarrow$ $\mathcal{F}$ is Gieseker stable $\Rightarrow$ $\mathcal{F}$ is Gieseker semistable $\Rightarrow$ $\mathcal{F}$ is μ-semistable.

Lemma 2.16. *Every Gieseker or μ-stable sheaf $\mathcal{F}$ is simple, i.e.* $\mathrm{Hom}(\mathcal{F}, \mathcal{F}) = \mathbb{C} \cdot \mathrm{id}_{\mathcal{F}}$.

Proof. [HL97, Cor. 1.2.8]. □

Remark 2.17. One can equivalently formulate the notion of stability in terms of destabilising quotients instead of subsheaves. Using this alternative description we easily find that for every locally free sheaf $\mathcal{F}$ we have

$$\mathcal{F} \text{ stable} \quad \Leftrightarrow \quad \mathcal{F}^\vee \text{ stable.}$$

Often this can simplify the proof of stability: Assume we have a rank three vector bundle $\mathcal{F}$ and we have proven that it does not contain destabilising subsheaves of rank one. If we can proof that the dual $\mathcal{F}^\vee$ does not contain rank one destabilising subsheaves neither, we can conclude that $\mathcal{F}$ is stable.

2.2 Moduli Spaces of Sheaves

The importance of the notion of Gieseker stability is due to the fact that there is a very general theory of the existence and compactness of moduli spaces of Gieseker semistable sheaves. First of all, we have to fix numerical invariants such as the rank and the Chern classes of the sheaves we want to parametrise. An elegant way to do so is using Mukai vectors:

Definition 2.18. Let X be a smooth projective varitey and $\mathcal{F}$ a sheaf on X. We call

$$v(\mathcal{F}) := \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_X} \in \mathrm{H}^*(X, \mathbb{Q})$$

the *Mukai vector of* $\mathcal{F}$.

Remark 2.19. Note that for any smooth projective variety X and any class $(1, c_1, c_2, \dots) \in \mathrm{H}^*(X, \mathbb{Q})$ we can formally define its square root (cf. [Cal05, p. 42]). If X is a $K3$ surface, we have $\mathrm{td}_X = (1, 0, 2)$ and hence $\sqrt{\mathrm{td}_X} = (1, 0, 1)$.

Now we fix an element $v \in \mathrm{H}^*(X, \mathbb{Q})$ and define the moduli functor.

Definition 2.20. For every scheme T we set

$$\mathcal{M}(v)(T) = \left\{ \begin{array}{c} \text{isomorphism classes of } T\text{-flat sheaves } \mathcal{F} \text{ on } X \times T \text{ such} \\ \text{that } \mathcal{F}_t \text{ is Gieseker semistable and } v(\mathcal{F}_t) = v \text{ for all } t \in T \end{array} \right\}.$$

This defines a contravariant functor $\mathcal{M}(v)(-)\colon (\textit{schemes}) \to (\textit{sets})$, where for every morphism $T \to T'$ the map $\mathcal{M}(v)(T') \to \mathcal{M}(v)(T)$ is obtained by pullback.

Theorem 2.21. *There is a projective scheme $M(v)$ universally corepresenting the functor $\mathcal{M}(v)(-)$. Furthermore, there are open subschemes $M^\mu(v) \subseteq M^s(v) \subseteq M(v)$ parametrising only μ-stable and Gieseker stable sheaves, respectively.*

Proof. [HL97, Thm. 4.3.4]. To prove that $M^\mu(v) \subseteq M(v)$ is open we can use a similar reasoning as in [HL97, Prop. 2.3.1]. □

The scheme above is called *moduli space of semistable sheaves on X with Mukai vector v* etc. The closed points of $M^\mu(v)$ and $M^s(v)$ are in one-to-one correspondence with slope or Gieseker stable sheaves on X with Mukai vector v. Note that in general $M(v)$ is far from being a fine moduli space. In fact, $M(v)$ parametrises so called *S-equivalence classes* (cf. [HL97, Sect. 1.5]).

Remark 2.22. Note that there is also a relative notion of the moduli functor and moduli spaces: In this case we consider a family $\mathcal{X} \to S$ of projective schemes together with a line bundle which is ample on every fibre $\mathcal{X}_s$. A Mukai vector is then an element $v \in \mathrm{H}^*(\mathcal{X}, \mathbb{Z})$. The pullback v_s of v to the fibres is, of course, locally constant. There is a relative moduli space $\mathcal{M}_S(v)$ parametrising sheaves on $\mathcal{X}$ such that the restriction to every fibre $\mathcal{X}_s$ is a stable sheaf with Mukai vector v_s (cf. [HL97, Thm. 4.3.7]).

Definition 2.23. If the moduli space $\mathcal{M}^s(v)$ represents the functor $\mathcal{M}^s(v)(-)$, we call it a *fine* moduli space.

If $\mathcal{M}^s(v)$ is fine, then it satisfies the follwing universal property: For every scheme T we have $\mathcal{M}^s(v)(T) = \mathcal{M}or(T, \mathcal{M}^s(v))$. For a family $\mathcal{G} \in \mathcal{M}^s(v)(T)$ we denote the induced morphism (also referred to as the *classifying map*) by $f_\mathcal{G}$.

Definition 2.24. A $\mathcal{M}^s(v)$-flat family on $X \times \mathcal{M}^s(v)$ is called *universal family* if for every scheme T, every family $\mathcal{G} \in \mathcal{M}(v)(T)$ and the classifying map $f_\mathcal{G} \colon T \to \mathcal{M}(v)$ there is a line bundle $\mathcal{L}$ on T such that $\mathcal{G} \otimes p_T^\star \mathcal{L} \simeq f_\mathcal{G}^\star \mathcal{E}$, where $p_T \colon X \times T \to T$ denotes the projection.

If it exists, a universal family is unique up to twist by a line bundle from $\mathcal{M}(v)$. If $\mathcal{F} \in \mathcal{M}^s(v)(\{pt\})$ is stable sheaf on X and $[\mathcal{F}] \in \mathcal{M}^s(v)$ the image of the classifying map, we have $\mathcal{E}|_{X \times [\mathcal{F}]} \simeq \mathcal{F}$. Unfortunately universal families do not exist in general. There is a weaker notion:

Definition 2.25. A $\mathcal{M}^s(v)$-flat family $\mathcal{E}$ on $X \times \mathcal{M}^s(v)$ is called *quasi-universal* if for every scheme T and every family $\mathcal{G} \in \mathcal{M}^s(v)(T)$ there is a classifying morphism $f_\mathcal{G} \colon T \to \mathcal{M}(v)$ and a locally free sheaf $\mathcal{W}$ on T such that $\mathcal{G} \otimes p_T^\star \mathcal{W} \simeq f_\mathcal{G}^\star \mathcal{E}$.

Proposition 4.6.2 in [HL97] states that there always exists a quasi-universal family. The rank of the vector bundles $\mathcal{W}$ in the above definition is constant. It is called the *multiplicity* of the quasi-universal family $\mathcal{E}$ and is usually denoted by σ. For a stable sheaf $\mathcal{F}$ on X we have $\mathcal{E}|_{X\times[\mathcal{F}]} \simeq \mathcal{F}^{\oplus\sigma}$.

The notion of stability heavily depends on the chosen polarisation H. Sometimes it is useful to emphasise this dependence. In this case we write $M_H(v)$ instead of $M(v)$.

In general the moduli spaces above may be highly singular. The local structure of the moduli space is closely related to the deformation properties of the sheaves it parametrises.

Proposition 2.26. *Let $\mathcal{F}$ be a Gieseker stable sheaf on X with Mukai vector v. Then the Zariski tangent space of $M(v)$ at the point $\mathcal{F}$ is given by $T_{\mathcal{F}}M(v) \cong \mathrm{Ext}^1(\mathcal{F},\mathcal{F})$. If $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})_0 = 0$, then $M(v)$ is smooth at $\mathcal{F}$.*

Proof. [HL97, Thm. 4.5.4]. □

Here we write $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})_0$ for the kernel of the natural trace map $\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \to \mathrm{H}^2(X,\mathcal{O}_X)$.

The deformation theory of strictly semistable sheaves is even more complicated.

2.3 Moduli Spaces of Sheaves on $K3$ Surfaces

If we restrict ourselves to the case where X is a projective $K3$ surface, we can continue the study of moduli spaces of sheaves in much more detail.

First of all, the Mukai vector of a sheaf $\mathcal{F}$ can be written as

$$v(\mathcal{F}) = (r,l,s) = \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_X} = (r, c_1, c_1^2/2 - c_2)(1,0,1) = (r, c_1, c_1^2/2 - c_2 + r).$$

This is an element in $\mathrm{H}^*(X,\mathbb{Z}) = \mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$. We can endow the latter with a lattice structure as follows:

$$(r_1,l_1,s_1)(r_2,l_2,s_2) := l_1 \cdot l_2 - r_1 s_2 - r_2 s_1.$$

We denote this lattice by $\widetilde{\mathrm{H}}(X,\mathbb{Z})$ and call it the *Mukai lattice of X*. The advantage of this notion is the following: Let $\mathcal{F}$ be a Gieseker stable sheaf on X with Mukai vector v. By Lemma 2.16 and Serre Duality we have $\dim \mathrm{Hom}(\mathcal{F},\mathcal{F}) = \dim \mathrm{Ext}^2(\mathcal{F},\mathcal{F}) = 1$. Using the Hirzebruch−Riemann−Roch theorem, we deduce

$$\dim T_{\mathcal{F}}M(v) = \dim \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) = v(\mathcal{F})^2 + 2.$$

Mukai also introduced a weight two Hodge structure on $\widetilde{\mathrm{H}}(X,\mathbb{Z})$:

$$\widetilde{\mathrm{H}}^{1,1}(X) := \mathrm{H}^0(X) \oplus \mathrm{H}^{1,1}(X) \oplus \mathrm{H}^4(X), \quad \widetilde{\mathrm{H}}^{0,2}(X) := \mathrm{H}^{0,2}(X), \quad \widetilde{\mathrm{H}}^{2,0}(X) := \mathrm{H}^{2,0}(X).$$

Next, we want to study the local structure of moduli spaces of sheaves on $K3$ surfaces. Since every Gieseker stable sheaf is simple, we can apply Serre duality to infer $\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \cong \mathrm{H}^2(X,\mathcal{O}_X) \cong \mathbb{C}$. Thus by Proposition 2.26 we find that $M^s(v)$ is smooth.

In general the moduli space $M(v)$ has singularities at points corresponding to strictly semistable sheaves. To avoid this issue there is a quite general concept to exclude the existence of strictly semistable sheaves at all. The main references are [HL97, Sect. 4.C] and [Yosh01, Sect. 1.4] in the case of torsion sheaves. There is also a well-written summary in the appendix of [Zow12]. Let us recall the most important aspects:

Definition 2.27. Let $\mathcal{F}$ be a sheaf on X of rank r and Chern classes c_i, $i = 1, 2$. We define the *discriminant of* $\mathcal{F}$ as

$$\Delta_{\mathcal{F}} := 2rc_2 - (r-1)c_1^2.$$

Certainly $\Delta_{\mathcal{F}}$ only depends on the Mukai vector $v(\mathcal{F})$. Thus we write Δ_v for $\Delta_{\mathcal{F}}$.

Definition 2.28. Fix a Mukai vector $v \in \mathrm{H}^*(X, \mathbb{Z})$. The set of *$v$-walls* is defined as

$$\Big\{\xi^{\perp} \cap \mathrm{Amp}(X)_{\mathbb{Q}} \mid \xi \in \mathrm{Num}(X) \text{ satisfying } \frac{r^2}{4}\Delta_v \leq \xi^2 < 0\Big\}.$$

A polarisation $H \in \mathrm{Amp}(X)$ is called *v-general* if it is not contained in a v-wall.

By Lemma 4.C.2 in [HL97] the set of v-walls is locally finite.

Lemma 2.29. *Let $v \in \mathrm{H}^*(X, \mathbb{Z})$ be a primitive Mukai vector. If the polarisation H is v-general, then there exist no strictly μ_H-semistable sheaves on X with Mukai vector v.*

Proof. [HL97, Thm. 4.C.3]. □

Thus we can deduce:

Proposition 2.30. *If v is primitive and the polarisation H is v-general, then the moduli space $M_H(v)$ is a smooth projective manifold.*

The following result due to Mukai is fundamental in the theory of moduli spaces of sheaves on $K3$s.

Theorem 2.31. *There is a non-degenerate symplectic structure on $M^s(v)$ which at a point corresponding to a Gieseker stable sheaf $\mathcal{F}$ coincides with the natural Yoneda pairing:*

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \times \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) \cong \mathbb{C}.$$

Proof. [Muk84, Thm. 0.1]. □

Corollary 2.32. *Let v be a primitive Mukai vector and H be v-general. Then $M_H(v)$ is a smooth symplectic manifold.*

Finally, we have the following famous theorem by O'Grady:

Theorem 2.33. *Let $l \in \mathrm{NS}\,X$ be a primitive class, $v = (r, l, s)$ a Mukai vector such that $r \geq 0$ and in the case $r = 0$, the class l is effective. Let H be a v-general polarisation. Then $M_H(v)$ is an irreducible symplectic manifold deformation equivalent to $X^{[n]}$ with $n = v^2/2 + 1$. Furthermore, there is an isomorphism of lattices and integral hodge structures*

$$\mathrm{H}^2(M_H(v)) \cong v^\perp \subset \widetilde{\mathrm{H}}(X, \mathbb{Z}),$$

where on the left hand side the lattice structure is given by the Beauville–Bogomolov pairing.

Proof. [HL97, Thm. 6.2.5], [O'G96]. □

Remark 2.34. There are some generalisations of this result to the case of non-primitive classes $l \in \mathrm{NS}\,X$ due to Yoshioka. The interested reader is directed to [Yosh99].

2.4 Induced Automorphisms on Moduli Spaces of Sheaves

In this section we discuss some general aspects of automorphisms on moduli spaces of sheaves on $K3$ surfaces which are induced by automorphisms of the underlying surface.

The following proposition on induced automorphisms on moduli spaces is the central result of this section. Though we are only interested in the case of moduli spaces on $K3$ surfaces we state it in a more general setting:

Proposition 2.35. *Let (X, H) be a polarised smooth projective variety and φ an automorphism of X preserving H. Consider a Mukai vector $v \in \mathrm{H}^*(X, \mathbb{Z})$ which is invariant under the induced action of φ. Then φ induces a biregular automorphism ι on $\mathcal{M}(v)$.*

Proof. Pointwise the automorphism ι should map a stable sheaf $\mathcal{F}$ to the pullback $\varphi^\star\mathcal{F}$. If $\mathcal{M}(v) = \mathcal{M}^s(v)$ is a fine moduli space, this assignment can be turned into a global morphism: Let $\mathcal{E}$ be a universal family on $X \times \mathcal{M}^s(v)$. We consider its pullback $(\varphi \times \mathrm{id}_{\mathcal{M}^s(v)})^\star\mathcal{E}$. This is a flat family of stable sheaves parametrised by $\mathcal{M}^s(v)$ and, by the universal property of $\mathcal{E}$ and $\mathcal{M}^s(v)$, we get a classifying morphism $\iota \colon \mathcal{M}^s(v) \to \mathcal{M}^s(v)$ satisfying $(\mathrm{id}_X \times \iota)^\star(\varphi \times \mathrm{id}_{\mathcal{M}^s(v)})^\star\mathcal{E} \simeq \mathcal{E}$.

In general a universal family does not exist but we can use the universal property of the moduli space $\mathcal{M}(v)$ to proceed nevertheless: By pullback, the automorphism φ induces an automorphism of the associated moduli functor. Since $\mathcal{M}(v)$ corepresents the moduli functor, we get the induced automorphism ι. □

Remark 2.36. The proposition above can certainly be generalised to the relative setting: Let $\mathcal{X} \to S$ be an S-flat family of smooth projective varieties together with an automorphism φ of $\mathcal{X}$ that acts fibrewise. Let $\mathcal{L}$ be a polarisation on $\mathcal{X}$ such that for every $s \in S$ the restriction $\mathcal{L}_s$ is preserved by the automorphism φ_s on the fibre $\mathcal{X}_s$. Furthermore let $v \in \mathrm{H}^*(\mathcal{X}, \mathbb{Z})$ be a Mukai vector, such that for all $s \in S$ the pullback v_s to the fibre $\mathcal{X}_s$ is φ_s-invariant. (It is enough to check this on one fibre.) Then φ induces a biregular automorphism ι on the relative moduli space $\mathcal{M}^s_S(v) \to S$ such that the restriction ι_s coincides with the induced automorphism on $(\mathcal{M}^s_S(v))_s$ from Proposition 2.35 above.

Remark 2.37. In the case that X is a $K3$ surface and if $\mathcal{M}^s(v)$ is compact and fine, any universal family $\mathcal{E}$ is simple: Let $p\colon X \times \mathcal{M}(v) \to \mathcal{M}^s(v)$ be the second projection. Since every stable sheaf is simple, the sheaf $p_\star \mathcal{H}om(\mathcal{E}, \mathcal{E})$ is a line bundle. It contains the structure sheaf which corresponds to scalar multiplications. A splitting of this inclusion is given by the trace map. Hence $p_\star \mathcal{H}om(\mathcal{E}, \mathcal{E}) \simeq \mathcal{O}$ and $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \cong \mathrm{H}^0(X \times \mathcal{M}^s(v), \mathcal{H}om(\mathcal{E}, \mathcal{E})) \cong \mathrm{H}^0(\mathcal{M}^s(v), p_\star \mathcal{H}om(\mathcal{E}, \mathcal{E})) \cong \mathbb{C}$.

Furthermore, if φ is an involution, $\mathcal{E}$ admits a linearisation with respect to the involution $(\varphi \times \iota)$: By the above considerations the sheaf $\mathcal{E}$ is $(\varphi \times \iota)$-invariant. Thus we have an isomorphism $f\colon (\varphi \times \iota)^\star \mathcal{E} \to \mathcal{E}$. Since $\mathcal{E}$ is simple, the square $f^2\colon \mathcal{E} = ((\varphi \times \iota)^\star)^2 \mathcal{E} \to \mathcal{E}$ is given by a scalar $\lambda \in \mathbb{C}^*$. Now we define $f_\lambda := \frac{1}{\sqrt{\lambda}} f$, for some root $\sqrt{\lambda}$. We have $f_\lambda^2 = \mathrm{id}_{\mathcal{E}}$. For a more general treatment we refer to Lemma 1 in [Plo07] and the remark thereafter.

Lemma 2.38. *O'Grady's isomorphism* $\mathrm{H}^2(\mathcal{M}(v), \mathbb{Z}) \cong v^\perp$ *of* Theorem 2.33 *is equivariant with respect to* φ *and* ι*. Thus we can compute the invariant lattice as*

$$\mathrm{H}^2(\mathcal{M}(v), \mathbb{Z})^\iota \cong (v^\perp)^\varphi.$$

Proof. This is an easy consequence of the construction of the above isomorphism: Denote by n the dimension of $\mathcal{M}(v)$. Let $\mathcal{E}$ be a quasi-universal sheaf on $X \times \mathcal{M}(v)$ with multiplicity σ and denote by $q\colon X \times \mathcal{M}(v) \to X$ and $p\colon X \times \mathcal{M}(v) \to \mathcal{M}(v)$ the natural projections. O'Grady defines a map

$$\begin{array}{rcl} \mathrm{H}^*(X, \mathbb{Z}) & \to & \mathrm{H}^2(\mathcal{M}(v), \mathbb{Z}) \\ \alpha & \mapsto & \dfrac{1}{\sigma} p_\star \big[q^\star \alpha \cdot \mathrm{ch}(\mathcal{E}) \cdot q^\star \sqrt{\mathrm{td}_X} \big]_3, \quad \alpha \in \mathrm{H}^*(X, \mathbb{Z}). \end{array}$$

Here the $[-]_3$ indicates the projection onto $\mathrm{H}^6(X \times \mathcal{M}(v), \mathbb{Z})$. Restricting this to $v^\perp$ yields the desired homomorphism which is independent of the choice of $\mathcal{E}$. But by definition $\mathcal{E}$ is $(\varphi \times \iota)$-invariant. Thus $\mathrm{ch}(\mathcal{E})$ is invariant, too. Hence the lemma. $\square$

The above observations lead to a quite general concept to construct and study automorphisms on moduli spaces of sheaves. This is applied in a very special situation in Section 7.2. Note that the method could be used in many other situations; for example, also for moduli spaces of sheaves on abelian surfaces.

3 Tautological Sheaves

One of the main results of this thesis concerns the stability of so-called *tautological sheaves.* In this chapter we collect the most important results concerning these sheaves.

3.1 Definition and Basic Results

Let X be a smooth projective surface and $\mathcal{F}$ a sheaf on X. Recall that there is the universal subscheme $\Xi_n \subset X \times X^{[n]}$ and we have the two projections $p_n \colon X \times X^{[n]} \to X^{[n]}$ and $q_n \colon X \times X^{[n]} \to X$.

Definition 3.1. The *tautological sheaf associated with* $\mathcal{F}$ is defined as

$$\mathcal{F}^{[n]} := p_{n\star}(q_n^\star \mathcal{F} \otimes \mathcal{O}_{\Xi_n}).$$

Remark 3.2. Very important for the study of tautological sheaves is the following observation: The universal subscheme Ξ_n and the nested Hilbert scheme $X^{[n-1,n]}$ are isomorphic outside codimension four subschemes (cf. Section 1.4). Let U denote the open subset where they are actually isomorphic. The restrictions of $q_n^\star \mathcal{F}$ and $\sigma_n^\star q_{n-1}^\star \mathcal{F}$ to U are naturally isomorphic. Thus the restriction of $\mathcal{F}^{[n]}$ to the image $p_n(U)$ in $X^{[n]}$ is isomorphic to $\widetilde{\mathcal{F}^{[n]}} := \psi_{n\star}\sigma_n^\star q_{n-1}^\star \mathcal{F}$ (restricted to $\psi_n(U) = p_n(U)$). Hence we can use $\widetilde{\mathcal{F}^{[n]}}$ instead of $\mathcal{F}^{[n]}$ as long as we want to study properties that are not sensible with respect to modifications in codimension four. In the case $n = 2$ we, in fact, have $\widetilde{\mathcal{F}^{[2]}} \simeq \mathcal{F}^{[2]}$.

The restriction of p_n to Ξ_n is a flat covering of degree n. Hence the following lemma:

Lemma 3.3. *If $\mathcal{F}$ is locally free (torsion-free, resp.), so is $\mathcal{F}^{[n]}$. If $\mathcal{F}$ has rank r, then $\mathcal{F}^{[n]}$ has rank nr.*

Proof. [Scal09b, Rem. 2.5 and Lem. 2.23]. □

Lemma 3.4. *Let $\mathcal{F}$ be a locally free sheaf on X. Then*

$$(\mathcal{F}^{[n]})^\vee \simeq (\mathcal{F}^\vee)^{[n]} \otimes \mathcal{O}(\delta_n).$$

Proof. Recall that $p_n^\star \mathcal{O}(\delta_n)$ is the relative canonical sheaf of the degree n covering p_n. Using Grothendieck–Verdier duality, we have

$$\begin{aligned} (\mathcal{F}^{[n]})^\vee &= \mathcal{H}om_{\mathcal{O}_{X^{[n]}}}(p_{n\star} q_n^\star \mathcal{F}, \mathcal{O}_{X^{[n]}}) &\simeq& \; p_{n\star}\mathcal{H}om_{\mathcal{O}_{\Xi_n}}(q_n^\star \mathcal{F}, p_n^\star \mathcal{O}(\delta_n)) \\ &\simeq p_{n\star}(q_n^\star \mathcal{F}^\vee \otimes p_n^\star \mathcal{O}(\delta_n)) &\simeq& \; (\mathcal{F}^\vee)^{[n]} \otimes \mathcal{O}(\delta_n). \end{aligned}$$ □

Lemma 3.5. *We have the following formula for the first Chern class of $\mathcal{F}^{[n]}$:*

$$c_1(\mathcal{F}^{[n]}) = c_1(\mathcal{F})_{X^{[n]}} - \mathrm{rk}(\mathcal{F})\delta_n.$$

Proof. The map $p_n\colon \Xi_n \to X^{[n]}$ is a flat covering of degree n with branch divisor δ_n. Thus the class of the relative canonical bundle of p_n is $p_n^\star\delta_n$. Hence the Grothendieck−Riemann−Roch theorem reads

$$\begin{aligned}
\mathrm{ch}(\mathcal{F}^{[n]}) \;=\; \mathrm{ch}(p_{n\star}q_n^\star\mathcal{F}) &= p_{n\star}(q_n^\star\,\mathrm{ch}(\mathcal{F})\cdot \mathrm{td}_{p_n}) \\
&= p_{n\star}((\mathrm{rk}(\mathcal{F}), q_n^\star c_1(\mathcal{F}),\dots)(1, -\tfrac{1}{2}p_n^\star\delta_n,\dots)) \\
&= p_{n\star}(\mathrm{rk}(\mathcal{F}), q_n^\star c_1(\mathcal{F}) - \tfrac{1}{2}\,\mathrm{rk}(\mathcal{F})p_n^\star\delta_n,\dots) \\
&= (n\,\mathrm{rk}(\mathcal{F}), p_{n\star}q_n^\star c_1(\mathcal{F}) - \mathrm{rk}(\mathcal{F})\delta_n,\dots).
\end{aligned}$$

Note that — as in the $n=2$ case — we have $p_{n\star}p_n^\star\delta_n = 2\delta_n$ because along the divisor $2\delta_n$ two sheets of the degree n covering come together.

Certainly the first Chern class is independent of modifications in codimenion four, i.e. $p_{n\star}q_n^\star c_1(\mathcal{F}) = \psi_{n\star}\sigma_n^\star q_{n-1}^\star c_1(\mathcal{F})$. Denote by $f_n\colon S^{n-1}X\times X \to S^nX$ the degree n covering and by $\mathrm{pr}_2\colon S^{n-1}X\times X \to X$ the second projection. We have $\psi_{n\star}\sigma_n^\star q_{n-1}^\star c_1(\mathcal{F}) = \rho_n^\star f_{n\star}\mathrm{pr}_2^\star c_1(\mathcal{F}) = c_1(\mathcal{F})_{X^{[n]}}$. □

3.2 Cohomology and Extension Groups

In this section we want to summarise the results of Scala and Krug about global sections and extensions of tautological sheaves. These formulas turn out to be a powerful tool to analyse stability and deformations of these sheaves. The work of Scala and Krug intensively uses the language of derived categories. Since in this thesis we are only interested in results concerning honest sheaves, we will concentrate on this case. Therefore we will not introduce the notion of derived categories but assume that the reader either is familiar with the basic concept of this field or is willing to skip the few results of this section where derived categories are mentioned. Throughout this section we consider a smooth quasi-projective surface X.

Let us start by recalling the fundamental result due to Bridgeland−King−Reid and Haiman concerning the derived category of coherent sheaves on the Hilbert scheme of points on a surface.

Theorem 3.6. *Let X be a smooth quasi-projective surface. We have an equivalence of categories*

$$\Phi\colon D^b(X^{[n]}) \to D^b_{\mathfrak{S}_n}(X^n)$$

called Bridgeland−King−Reid−Haiman *or* BKRH correspondence, *where on the right hand side we consider the $\mathfrak{S}_n$-equivariant bounded derived category on X^n.*

The proof of Theorem 3.6 is a combination of [BKR01] and [Hai01]. For a more detailed account see Section 1.5 in [Scal09a].

Scala computed the image $\Phi(\mathcal{F}^{[n]})$ of tautological sheaves under the BKRH correspondence. The exact statement of Scala's result contains a lot of combinatorial notation.

Let us only state the result for $n = 2$. We will also completely omit the linearisation of $\Phi(\mathcal{F}^{[n]})$ which of course is very important for the detailed computations of Scala and Krug. For $n = 2$ the equivalence Φ is nothing but $R\sigma_\star \circ \psi^\star$ in the notation of Section 1.3 (note that ψ is flat and need not be derived). Thus the description of $\Phi(\mathcal{F}^{[2]})$ is an easy geometric consideration which already appeared in [Dan01]:

Proposition 3.7. *We have exact sequences*

$$0 \to \psi^\star \mathcal{F}^{[2]} \to \sigma^\star \mathcal{F}^{\boxplus 2} \to i_\star \sigma_D^\star \mathcal{F} \to 0 \quad \text{on } \widetilde{X \times X} \text{ and}$$
$$0 \to \Phi(\mathcal{F}^{[2]}) \to \mathcal{F}^{\boxplus 2} \to \Delta_\star \mathcal{F} \to 0 \qquad \text{on } X \times X.$$

From the description of $\Phi(\mathcal{F}^{[n]})$ Scala deduced the following formula for the cohomology of tautological sheaves:

Theorem 3.8. *For every sheaf $\mathcal{F}$ and every line bundle $\mathcal{L}$ on X we have*

$$\mathrm{H}^*(X^{[n]}, \mathcal{F}^{[n]} \otimes \mathcal{L}_{X^{[n]}}) \cong \mathrm{H}^*(X, \mathcal{F} \otimes \mathcal{L}) \otimes S^{n-1}\, \mathrm{H}^*(X, \mathcal{L}).$$

Proof. [Scal09b, Cor. 4.5], [Kru11, Thm. 6.17]. □

We continue by stating Krug's formula for the extension groups of tautological sheaves:

Theorem 3.9. *Let $\mathcal{F}$ and $\mathcal{E}$ be sheaves and $\mathcal{L}$ and $\mathcal{M}$ be line bundles on X. We have*

$$\mathrm{Ext}^*_{X^{[n]}}(\mathcal{E}^{[n]} \otimes \mathcal{L}_{X^{[n]}}, \mathcal{F}^{[n]} \otimes \mathcal{M}_{X^{[n]}}) \cong \begin{array}{c} \mathrm{Ext}^*_X(\mathcal{E} \otimes \mathcal{L}, \mathcal{F} \otimes \mathcal{M}) \otimes S^{n-1}\, \mathrm{Ext}^*_X(\mathcal{L}, \mathcal{M}) \\ \bigoplus \\ \mathrm{Ext}^*_X(\mathcal{E} \otimes \mathcal{L}, \mathcal{M}) \otimes \mathrm{Ext}^*_X(\mathcal{L}, \mathcal{F} \otimes \mathcal{M}) \otimes \\ S^{n-2}\, \mathrm{Ext}^*_X(\mathcal{L}, \mathcal{M}). \end{array} \tag{7}$$

Proof. [Kru11, Thm. 6.17]. □

Krug also gave a description how to compute Yoneda products on these extension groups (cf. [Kru11, Sect. 7]). The general formulas are extremely long. We will give a more detailed account on them as needed.

Let us finish this section by deriving a special case of formula (7).

Corollary 3.10. *Let X be a $K3$ surface and let $\mathcal{F}$ be a sheaf on X satisfying $h^2(\mathcal{F}) = 0$. Then we have*

$$\begin{aligned} \mathrm{Hom}_{X^{[n]}}(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}) &\cong \mathrm{Hom}_X(\mathcal{F}, \mathcal{F}), \\ \mathrm{Ext}^1_{X^{[n]}}(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}) &\cong \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F}) \bigoplus \mathrm{H}^0(X, \mathcal{F}) \otimes \mathrm{H}^1(X, \mathcal{F})^\vee. \end{aligned} \tag{8}$$

Remark 3.11. From these equations we can deduce that tautological sheaves $\mathcal{F}^{[2]}$ associated with stable sheaves $\mathcal{F} \not\cong \mathcal{O}_X$ are always simple: By Serre Duality a stable sheaf $\mathcal{F} \not\cong \mathcal{O}_X$ on a $K3$ surface satisfies either $h^2(\mathcal{F}) = 0$ or $h^0(\mathcal{F}) = 0$ and by twisting with a suitable line bundle we may assume that $h^2(\mathcal{F}) = 0$. This is a first indication that tautological sheaves might be stable.

3.3 Polarisations and slopes

In this section we shall talk about polarisations on the Hilbert scheme of points on a surface. In general the ample cone of these varieties is not completely known. Nevertheless, if we fix a polarisation H on our surface X, we will define polarisations H_N on $X^{[n]}$, depending on H and an integer N. Furthermore, we shall derive and discuss the slopes of tautological sheaves with respect to these polarisations. This will be important when we want to study the stability of these sheaves in Chapters 4 and 5.

Fix a smooth projective surface X and an ample class $H \in \operatorname{NS} X$. For any integer N we consider the class

$$H_N := NH_{X^{[n]}} - \delta_n \in \operatorname{NS} X^{[n]}.$$

Lemma 3.12. *For all sufficiently large N, the class H_N is ample.*

Proof. For $n = 2$ the Hilbert−Chow morphism ρ is a blow up. The class $H_{X^{[2]}}$ is the pullback of an ample class on S^2X and $-\delta$ is ample on the fibres of ρ. For $n > 2$ we proceed by induction. By Corollary 1.19 and Lemma 1.20 we have

$$\psi_n^\star H_N = \sigma_n^\star(p_{n-1}^\star(NH_{X^{[n-1]}} - \delta_{n-1}) + Nq_{n-1}^\star H) - [D_n].$$

By induction $NH_{X^{[n-1]}} - \delta_{n-1}$ is ample on $X^{[n-1]}$. Hence for sufficiently large N, $\psi_n^\star H_N$ is ample. By [Laz04, Cor. 1.2.24] H_N is ample, too. □

In order to compute slopes of tautological sheaves, we need to compute intersection numbers. We have the following general result:

Lemma 3.13. *Let l be a class in* $\operatorname{NS} X$*. We have*

$$l_{X^{[n]}}.H_{X^{[n]}}^{2n-1} = \frac{n}{2^{n-1}}(l.H)(H^2)^{n-1} \text{ and} \tag{9}$$

$$\delta_n.H_{X^{[n]}}^{2n-1} = 0, \tag{10}$$

where on the right hand side of (9) *we consider the intersection in* $\operatorname{NS} X$*.*

Proof. Both $l_{X^{[n]}}$ and $H_{X^{[n]}}$ are pullbacks from S^nX along the Hilbert−Chow morphism. We pull back along the $n!$-fold covering $X^n \to S^nX$ and obtain the classes $l^{\boxplus n}$ and $H^{\boxplus n}$, respectively. We have

$$l_{X^{[n]}}.H_{X^{[n]}}^{2n-1} = \frac{1}{n!}(l^{\boxplus n})(H^{\boxplus n})^{2n-1} = \frac{1}{n!}\binom{n}{1,2,\ldots,2}n(l.H)(H^2)^{n-1} = \frac{n}{2^{n-1}}(l.H)(H^2)^{n-1}.$$

In order to prove (10), it is certainly enough to show that

$$(\psi_n^\star \delta_n).(\psi_n^\star H_{X^{[n]}}^{2n-1}) = 0. \tag{11}$$

We will use an induction argument. For $n = 2$, equation (11) reads

$$D.\sigma^\star (H^{\boxplus 2})^3 = 0.$$

This is true by Lemma 1.15c). Now for the induction step we use Lemmata 1.18 and 1.20:

$$\begin{aligned}
& (\psi_n^\star \delta_n).(\psi_n^\star H_{X^{[n]}}^{2n-1}) \\
= \;& \sigma_n^\star\big((p_{n-1}^\star H_{X^{[n-1]}} + q_{n-1}^\star H)^{2n-1} p_{n-1}^\star \delta_{n-1}\big) \quad + \quad \underbrace{\sigma_n^\star (p_{n-1}^\star H_{X^{[n-1]}} + q_{n-1}^\star H)^{2n-1}[D_n]}_{\|} \\
= \;& \tbinom{2n-1}{2} p_{n-1}^\star (\delta_{n-1}.H^{2n-3}) q_{n-1}^\star H^2 \quad + \quad (\qquad \textit{as above} \qquad).
\end{aligned}$$

Now by induction the first term vanishes. And for the second term we can apply exactly the same reasoning as in Lemma 1.15c). □

Corollary 3.14. *Let $\mathcal{L}$ be a line bundle on X with first Chern class l and $\mathcal{F}$ a sheaf of rank r and first Chern class f. We have the following expansions for the slopes of $\mathcal{F}^{[n]}$ and $\mathcal{L}$ with respect to H_N:*

$$\begin{aligned}
\mu_{H_N}(\mathcal{L}_{X^{[n]}}) &= N^{2n-1}\frac{n}{2^{n-1}}(l.H)(H^2)^{n-1} + O(N^{2n-2}) \quad \textit{and} \\
\mu_{H_N}(\mathcal{F}^{[n]}) &= N^{2n-1}\frac{n}{2^{n-1}}\frac{1}{nr}(f.H)(H^2)^{n-1} + O(N^{2n-2}).
\end{aligned}$$

If $X = A$ is an abelian surface, there is another candidate for a polarisation. Recall that we have a summation morphism $m_n \colon A^{[n]} \to A$ and at least one additional summand in NS $A^{[n]}$ containing $(\mathrm{NS}\, A)_{M_n}$. As will be explained in Lemma 5.7, classes in $(\mathrm{NS}\, A)_{M_n}$ have degree zero with repect to the polarisation H_N, which turns out to be inconvenient for the proof of stability of tautological sheaves. To circumvent this issue we will consider the following polarisation:

Lemma 3.15. *For all $N \gg 0$ the class*

$$H_N^m := NH_{X^{[n]}} - \delta_n + Nm_n^\star H$$

is ample.

Proof. By [Laz04, Exa. 1.4.4] the pullback $m_n^\star H$ is nef. Hence we can apply [Laz04, Cor. 1.4.10] to conclude that H_N^m is ample. □

4 Stability of Tautological Sheaves: Regular Surfaces

In this chapter we prove the stability of rank two and rank four tautological sheaves on the Hilbert scheme of two points and of rank three tautological sheaves on the Hilbert scheme of three points on a regular surface. Thus fix a smooth projective surface X satisfying $h^1(X, \mathcal{O}_X) = 0$ together with an ample class $H \in \operatorname{NS} X$. We will use the terms 'stability', 'stable', 'semistable' etc. to denote μ-stability, μ-stable and so on.

4.1 Destabilising Line Subbundles

In this section we will show that for $N \gg 0$ there exist no H_N-destabilising line subbundles $\mathcal{L} \subseteq \mathcal{F}^{[2]}$ in the case $\mathcal{F} \not\simeq \mathcal{O}_X$. So assume that $\mathcal{L}$ was such a destabilising line subbundle. Hence there is a non-trivial homomorphism in $\operatorname{Hom}(\mathcal{L}, \mathcal{F}^{[2]})$. Now we have $\mathcal{F}^{[2]} \simeq \psi_\star r_1^\star \mathcal{F}$. Using adjunction, we get

$$\operatorname{Hom}(\mathcal{L}, \mathcal{F}^{[2]}) \cong \operatorname{Hom}(\mathcal{L}, \psi_\star r_1^\star \mathcal{F}) \cong \operatorname{Hom}(\psi^\star \mathcal{L}, r_1^\star \mathcal{F}).$$

We can write $\psi^\star \mathcal{L} \simeq r_1^\star \mathcal{L}_1 \otimes r_2^\star \mathcal{L}_2 \otimes \mathcal{O}(aD)$ for some line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X with Chern classes l_1 and l_2, respectively. The line bundle $\psi^\star \mathcal{L}$ is $\mathfrak{S}_2$-invariant since it comes from $X^{[2]}$. Thus, in fact, we can assume that $\mathcal{L}_1 \simeq \mathcal{L}_2$ but for later use we will proceed in this generality. We have the following result:

Lemma 4.1. *Let $\mathcal{G}$ and $\mathcal{H}$ be locally free sheaves on $X \times X$ and $a \in \mathbb{Z}$. We have*

$$\begin{aligned} \operatorname{Hom}_{\widetilde{X \times X}}(\sigma^\star \mathcal{G} \otimes \mathcal{O}(aD), \sigma^\star \mathcal{H}) \quad &\subseteq \quad \operatorname{Hom}_{\widetilde{X \times X}}(\sigma^\star \mathcal{G}, \sigma^\star \mathcal{H}) \\ &\big(\cong \quad \operatorname{Hom}_{X \times X}(\mathcal{G}, \mathcal{H}) \big). \end{aligned}$$

Proof. Consider the ideal sheaf sequence of the exceptional divisor D:

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_{\widetilde{X \times X}} \to \mathcal{O}_D \to 0.$$

Tensoring this sequence with $\mathcal{O}(aD)$, we have

$$0 \to \mathcal{O}((a-1)D) \to \mathcal{O}(aD) \to \mathcal{O}_D(-a) \to 0. \tag{12}$$

We apply $\sigma_\star$. For $a = 0$ we have $\sigma_\star \mathcal{O}_{\widetilde{X \times X}} \simeq \mathcal{O}_{X \times X}$ and for $a < 0$ sequence (12) gives an inclusion $\sigma_\star \mathcal{O}(aD) \subseteq \mathcal{O}_{X \times X}$. For $a > 0$ we observe that the restriction of $\mathcal{O}_D(-a)$ to a fibre is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-a)$ which does not have global sections. Thus again by (12) $\sigma_\star \mathcal{O}((a-1)D) \simeq \sigma_\star \mathcal{O}(aD)$. Altogether we see that $\sigma_\star \mathcal{O}(aD)$ is contained in $\mathcal{O}_{X \times X}$ for all $a \in \mathbb{Z}$. We use adjunction $\sigma^\star \dashv \sigma_\star$to proceed:

$$\begin{aligned} \operatorname{Hom}(\sigma^\star \mathcal{G} \otimes \mathcal{O}(aD), \sigma^\star \mathcal{H}) \quad &\cong \quad \operatorname{Hom}(\sigma^\star \mathcal{G}, \sigma^\star \mathcal{H} \otimes \mathcal{O}(-aD)) \\ \cong \quad \operatorname{Hom}(\mathcal{G}, \mathcal{H} \otimes \sigma_\star \mathcal{O}(-aD)) \quad &\subseteq \quad \operatorname{Hom}(\mathcal{G}, \mathcal{H}). \end{aligned}$$

Note that since $\sigma_\star \mathcal{O}_{\widetilde{X \times X}} \simeq \mathcal{O}_{X \times X}$, the projection formula yields $\sigma_\star \circ \sigma^\star \simeq \mathrm{id}$. $\square$

Lemma 4.2. *For every locally free sheaf $\mathcal{F}$ and line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X we have*

$$\operatorname{Hom}_{\widetilde{X\times X}}(r_1^\star\mathcal{L}_1\otimes r_2^\star\mathcal{L}_2\otimes\mathcal{O}(aD),r_1^\star\mathcal{F}) \;\subseteq\; \operatorname{Hom}_X(\mathcal{L}_1,\mathcal{F})\otimes \mathrm{H}^0(X,\mathcal{L}_2^\vee).$$

Proof. Again we use adjunction, the Lemma above and the Künneth formula:

$$\begin{aligned}
&\operatorname{Hom}_{\widetilde{X\times X}}(r_1^\star\mathcal{L}_1\otimes r_2^\star\mathcal{L}_2\otimes\mathcal{O}(aD),r_1^\star\mathcal{F}) &&\subseteq \operatorname{Hom}_{\widetilde{X\times X}}(r_1^\star\mathcal{L}_1\otimes r_2^\star\mathcal{L}_2,r_1^\star\mathcal{F})\\
\cong\;&\operatorname{Hom}_{X\times X}(\pi_1^\star\mathcal{L}_1\otimes \pi_2^\star\mathcal{L}_2,\pi_1^\star\mathcal{F}) &&\cong \operatorname{Hom}_X(\mathcal{L}_1,\mathcal{F})\otimes \mathrm{H}^0(X,\mathcal{L}_2^\vee).
\end{aligned}$$

□

Corollary 4.3. *Let $\mathcal{F}$ be a μ_H-stable vector bundle on X of rank r and first Chern class $c_1(\mathcal{F})=f$. Then $r_1^\star\mathcal{F}$ contains no line subbundles $r_1^\star\mathcal{L}_1\otimes r_2^\star\mathcal{L}_2\otimes\mathcal{O}(aD)$ satisfying:*

$$H.(l_1+l_2)\geq\frac{H.f}{r},$$

except the case $r=1$, $\mathcal{L}_2\simeq\mathcal{O}_X$ and $\mathcal{L}_1\simeq\mathcal{F}$.

Proof. Let $r_1^\star\mathcal{L}_1\otimes r_2^\star\mathcal{L}_2\otimes\mathcal{O}(aD)$ be a line subbundle of $r_1^\star\mathcal{F}$ satisfying the hypothesis of the corollary. We will show that $\operatorname{Hom}_X(\mathcal{L}_1,\mathcal{F})\otimes\mathrm{H}^0(X,\mathcal{L}_2^\vee)=0$ which yields a contradiction to Lemma 4.2 above.

If $H.l_2>0$, we have $0=\mathrm{H}^0(X,\mathcal{L}_2^\vee)$ and we are done. If $H.l_2\leq 0$, we see

$$H.l_1\geq H.(l_1+l_2)\geq\frac{H.f}{r}. \tag{13}$$

Hence if $\mathcal{L}_1\not\simeq\mathcal{F}$, by the stability of $\mathcal{F}$ we have $\operatorname{Hom}_X(\mathcal{L}_1,\mathcal{F})=0$.

If $\mathcal{L}_1\simeq\mathcal{F}$, we must have $r=1$ and equalities everywhere in equation (13), thus $H.l_2=0$. But then again $\mathrm{H}^0(X,\mathcal{L}_2^\vee)=0$ for all $\mathcal{L}_2$ but the trivial line bundle. □

Since we are considering regular surfaces, all line bundles on $\widetilde{X\times X}$ are of the form $r_1^\star\mathcal{L}_1\otimes r_2^\star\mathcal{L}_2\otimes\mathcal{O}(aD)$ as in Corrolary 4.3 above (cf. Section 1.3).

Theorem 4.4. *Let $\mathcal{F}$ be a μ_H-stable vector bundle on X of rank r and first Chern class $c_1(\mathcal{F})=f$, Assume $\mathcal{F}\not\simeq\mathcal{O}_X$, Then for sufficiently large N, the tautological vector bundle $\mathcal{F}^{[2]}$ on $X^{[2]}$ does not contain any μ_{H_N}-destabilising line subbundles.*

Proof. Let $\mathcal{L}_{X^{[2]}}\otimes\mathcal{O}(a\delta)\subseteq\mathcal{F}^{[2]}$, ($\mathcal{L}\in\operatorname{Pic}X$) be a destabilising line subbundle. As explained at the beginning of this section, this line bundle yields a homomorphism $\operatorname{Hom}_{\widetilde{X\times X}}(\mathcal{L}^{\boxtimes 2}\otimes\mathcal{O}(aD),r_1^\star\mathcal{F})$. Let $l:=c_1(\mathcal{L})$. By Corollary 3.14 for all $N\gg 0$ the destabilising condition implies

$$H.l\geq\frac{H.f}{2r}.$$

This is clearly a contradiction to Corollary 4.3. □

4.2 The Cases $r = 1$ and $r = 2$

From Theorem 4.4 we deduce:

Corollary 4.5. *Let $\mathcal{F}$ be a line bundle on X not isomorphic to $\mathcal{O}_X$. Then for N sufficiently large, $\mathcal{F}^{[2]}$ is a μ_{H_N}-stable rank two vector bundle on $X^{[2]}$.*

Proof. Since $\mathcal{F}^{[2]}$ has rank two, we only have to consider torsion-free destabilising subsheaves of rank one. If $\mathcal{E}$ is such a subsheaf, we can embed it into its reflexive hull $\mathcal{E}^{\vee\vee}$. This is a reflexive rank one sheaf, i.e. a line bundle. Since $\mathcal{F}^{[2]}$ is locally free, it is also reflexive. Thus $\mathcal{E}^{\vee\vee}$ is a subbundle of $\mathcal{F}^{[2]}$. The first Chern classes of $\mathcal{E}^{\vee\vee}$ and $\mathcal{E}$ coincide, therefore $\mathcal{E}^{\vee\vee}$ is destabilising. This gives a contradiction to Theorem 4.4. □

We can generalise this result to arbitrary torsion-free rank one sheaves on X with nonvanishing first Chern class:

Theorem 4.6. *Let $\mathcal{F}$ be a rank one torsion-free sheaf on X satisfying $\det \mathcal{F} \not\simeq \mathcal{O}_X$. Then for N sufficiently large, $\mathcal{F}^{[2]}$ is a μ_{H_N}-stable rank two torsion-free sheaf on $X^{[2]}$.*

Proof. Every torsion-free rank one sheaf $\mathcal{F}$ on a smooth surface can be written as $\mathcal{F} \simeq \mathcal{L} \otimes \mathcal{I}_Z$ for some line bundle $\mathcal{L}$ and an ideal sheaf $\mathcal{I}_Z$ of a zero dimensional subscheme $Z \subset X$. We thus have an injection $\mathcal{F} \subseteq \mathcal{L}$ and, of course, $c_1(\mathcal{F}) = c_1(\mathcal{L})$.

Now, since $(-)^{[2]}$ is an exact functor (cf. [Scal09b, Lem. 2.23]), we have $\mathcal{F}^{[2]} \subseteq \mathcal{L}^{[2]}$. And since $\mathcal{L}^{[2]}$ is torsion-free, so is $\mathcal{F}^{[2]}$. Furthermore, $c_1(\mathcal{F}^{[2]}) = c_1(\mathcal{L}^{[2]})$ because the cokernel of the inclusion $\mathcal{F}^{[2]} \hookrightarrow \mathcal{L}^{[2]}$ is $\mathcal{O}_Z^{[2]}$ which is supported in codimension two. Hence the stability of $\mathcal{F}^{[2]}$ follows immediately from Corollary 4.5. □

Next, we want to consider the case $r = \operatorname{rk} \mathcal{F} = 2$. We have seen before that $\mathcal{F}^{[2]}$ cannot contain destabilising line subbundles. In this section we will prove that in most cases, in fact, $\mathcal{F}^{[2]}$ does not contain any destabilising subsheaves. We start with a technical lemma that we will need in the proof.

Lemma 4.7. *Let V be a smooth projective variety and let $i \colon Y \hookrightarrow V$ be a smooth divisor. For a rank r sheaf $\mathcal{E}$ on Y we have*

$$c_1(i_\star \mathcal{E}) = rY.$$

Proof. This follows easily from the Grothendieck−Riemann−Roch theorem:

$$\operatorname{ch}(i_\star \mathcal{E}) = i_\star(\operatorname{ch}(\mathcal{E}) \operatorname{td}_i) = i_\star((r + \dots)(1 + \dots)) = i_\star(r + \dots) = rY + \dots .$$

Here td_i denotes the Todd class of the relative tangent bundle of the embedding i. □

Theorem 4.8. *Let $\mathcal{F}$ be a rank two μ_H-stable sheaf on X with $c_1(\mathcal{F}) = f$. Assume $\det \mathcal{F} \not\simeq \mathcal{O}_X$. Then for N sufficiently large, $\mathcal{F}^{[2]}$ is a μ_{H_N}-stable rank four sheaf on $X^{[2]}$.*

Proof. First assume that $\mathcal{F}$ is reflexive, i.e. locally free. Let $\mathcal{E}$ be a destabilising subsheaf of $\mathcal{F}^{[2]}$. Write its first Chern class as $e_{X^{[2]}} + a\delta$, $e \in \mathrm{NS}\, X$, $a \in \mathbb{Z}$. We may assume that $\mathcal{E}$ is semistable. Similarly to the proof of Corollary 4.5 we can reduce to the case that $\mathcal{E}$ is reflexive and by Lemma 2.10 it is saturated. By Theorem 4.4, $\mathcal{E}$ cannot have rank one. So let us first consider the case $\mathrm{rk}\, \mathcal{E} = 3$ and let us have a look at the corresponding short exact sequence on $X^{[2]}$:

$$0 \to \mathcal{E} \to \mathcal{F}^{[2]} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the corresponding destabilising quotient of rank one. Using Lemma 3.4, we see that the dual of this sequence looks as follows:

$$0 \to \mathcal{Q}^{\vee} \to (\mathcal{F}^{\vee})^{[2]} \otimes \mathcal{O}(\delta) \to \mathcal{E}^{\vee} \to \mathcal{E}xt^1_{\mathcal{O}_{X^{[2]}}}(\mathcal{Q}, \mathcal{O}_{X^{[2]}}) \to 0.$$

Note that $\mathcal{E}xt^1_{\mathcal{O}_{X^{[2]}}}(\mathcal{F}^{[2]}, \mathcal{O}_{X^{[2]}})$ vanishes because $\mathcal{F}^{[2]}$ is locally free. Since $\mathcal{E}$ is saturated, $\mathcal{Q}$ is torsion-free and hence the support of $\mathcal{E}xt^1_{\mathcal{O}_{X^{[2]}}}(\mathcal{Q}, \mathcal{O}_{X^{[2]}})$ has codimension at least two, thus vanishing first Chern class. We may assume that $\mathcal{Q}^{\vee}$ is reflexive, i.e. locally free (it is of rank one). (If necessary, we replace $\mathcal{Q}^{\vee}$ by its reflexive hull which still gives a subsheaf of $(\mathcal{F}^{\vee})^{[2]} \otimes \mathcal{O}(\delta)$ with the same first Chern class.) We have an inclusion $\mathcal{Q}^{\vee} \otimes \mathcal{O}(-\delta) \hookrightarrow (\mathcal{F}^{\vee})^{[2]}$ and we compute its first Chern class as

$$\begin{aligned} c_1(\mathcal{Q}^{\vee} \otimes \mathcal{O}(-\delta)) &= -(c_1(\mathcal{F}^{[2]}) - c_1(\mathcal{E})) - \delta = -(f_{X^{[2]}} - e_{X^{[2]}} - a\delta) - \delta \\ &= e_{X^{[2]}} - f_{X^{[2]}} + (a-1)\delta. \end{aligned}$$

Now, by Corollary 3.14, the destabilising condition on $\mathcal{E}$ reads:

$$\mu_{H_N}(\mathcal{E}) = \frac{H.e}{3} N^3 H^2 + O(N^2) \geq \frac{H.f}{4} N^3 H^2 + O(N^2) = \mu_{H_N}(\mathcal{F}^{[2]}).$$

Thus $H.2(e-f) \geq -\frac{H.f}{2}$ and by Corollary 4.3 the line bundle $\mathcal{Q}^{\vee} \otimes \mathcal{O}(-\delta)$ does not admit a homomorphism to $(\mathcal{F}^{\vee})^{[2]}$. A contradiction. Note that the above considerations are a special instance of the more general fact that a sheaf is stable if and only if its dual is stable.

Finally, assume that the maximal destabilising subsheaf of $\mathcal{F}^{[2]}$ is a rank two sheaf $\mathcal{E}$. By using adjunction $\psi^{\star} \dashv \psi_{\star}$, we get a homomorphism $\beta\colon \psi^{\star}\mathcal{E} \to r_1^{\star}\mathcal{F}$. Now we will distinguish three cases:

a) $\mathrm{rk}\ker\beta = 0$.

Thus $\ker\beta$ is a torsion subsheaf of $\mathcal{E}$, so it is trivial since $\mathcal{E}$ is torsion-free. Hence β is an isomorphism outside an effective divisor $j\colon Y \hookrightarrow \widetilde{X \times X}$. Thus $\mathrm{coker}\,\beta$ can be written as $j_{\star}\mathcal{K}$ for some sheaf $\mathcal{K}$ on Y. Let $Y = \bigcup_i Y_i$ be the decomposition into irreducible components, then by Lemma 4.7 we can write its first Chern class as $c_1(\mathrm{coker}\,\beta) = \sum_i (Y_i \cdot \mathrm{rk}\,\mathcal{K}_i)$, where $\mathcal{K}_i$ is the restriction of $\mathcal{K}$ to Y_i. On the other hand we can compute the first Chern class of $\mathrm{coker}\,\beta$ directly:

$$c_1(\mathrm{coker}\,\beta) = c_1(r_1^{\star}\mathcal{F}) - c_1(\mathcal{E}) = f \otimes 1 - e \otimes 1 - 1 \otimes e - aD.$$

Now Y is effective. Thus if $\operatorname{rk}\mathcal{K}_i \neq 0$ for some i, the class $(f-e)\otimes 1 - 1\otimes e - aD$ must contain an effective divisor. Since D is the exceptional divisor of the blowup σ, this class is effective if and only if either $a < 0$ and $f = e = 0$ (which we excluded) or $a \leq 0$ and $(f-e)\otimes 1 - 1\otimes e$ is effective and nonzero. Evaluating against the polarisation $\psi^\star H_N$ implies $2H.e < H.f$. Together with the destabilising condition on $\mathcal{E}$ — which implies $2H.e \geq H.f$ — we get a contradiction. If $\operatorname{rk}\mathcal{K}_i = 0\ \forall i$, i.e. $c_1(\operatorname{coker}\beta) = 0$, we must have $f = 0$ which we excluded.

b) $\operatorname{rk}\ker\beta = 2$.
This says that β has to vanish on an open subset which contradicts the fact that $\mathcal{E}$ injects into $\mathcal{F}^{[2]}$.

c) $\operatorname{rk}\ker\beta = 1$.
Now $\operatorname{im}\beta$ is a rank one quotient sheaf of $\psi^\star\mathcal{E}$ and we write its first Chern class $c_1(\operatorname{im}\beta) = l_1\otimes 1 + 1\otimes l_2 + bD$. The semistability of $\mathcal{E}$ yields

$$H.e \leq H.(l_1 + l_2).$$

At the same time $\operatorname{im}\beta$ is a rank one subsheaf of $r_1^\star\mathcal{F}$. Denote by $\operatorname{im}\beta^{\vee\vee}$ its reflexive hull. This is a reflexive rank one sheaf, thus a line bundle. And it has the same first Chern class as $\operatorname{im}\beta$. The destabilising condition on $\mathcal{E}$ implies

$$2H.e \geq H.f.$$

Putting things together, we find a line subbundle in $r_1^\star\mathcal{F}$ satisfying

$$2H.(l_1 + l_2) \geq H.f.$$

This is a contradiction to Corollary 4.3.

Finally, if $\mathcal{F}$ is not locally free, we can embed it into its (locally free) reflexive hull $\mathcal{F}^{\vee\vee}$ and proceed exactly as in the proof of Theorem 4.6. □

4.3 Higher n

In this section we try to generalise the results on destabilising line subbundles in Section 4.1 to higher n. From this generalisation we will be able to prove the stability of rank three tautological sheaves on $X^{[3]}$.

Let $\mathcal{F}$ be a torsion-free μ_H-stable sheaf on X. Denote its rank by r and its first Chern class by f. We want to show that the associated tautological sheaf $\mathcal{F}^{[n]}$ on $X^{[n]}$ has no destabilising subsheaves of rank one. We will first assume that $\mathcal{F}$ is reflexive, i.e. locally free. Thus we may assume that a destabilising rank one subsheaf of $\mathcal{F}^{[n]}$ is also reflexive, that is, a line bundle.

Proposition 4.9. *For sufficiently large N, there are no μ_{H_N}-destabilising line subbundles in $\mathcal{F}^{[n]}$ of the form $\mathcal{L}_{X^{[n]}}$, ($\mathcal{L} \in \operatorname{Pic} X$), except the case $r = 1$ and $\mathcal{L} \simeq \mathcal{F} \simeq \mathcal{O}_X$.*

Proof. Denote the first Chern class of $\mathcal{L}$ by l. Using Scala's calculations of cohomology groups of tautological sheaves with twists as stated in Theorem 3.8 we can immediately deduce the following formula for homomorphisms from line bundles of the form $\mathcal{L}_{X^{[n]}}$ to tautological sheaves $\mathcal{F}^{[n]}$:

$$\mathrm{Hom}_{X^{[n]}}(\mathcal{L}_{X^{[n]}}, \mathcal{F}^{[n]}) \cong \mathrm{Hom}_X(\mathcal{L}, \mathcal{F}) \otimes \mathrm{Hom}_X(\mathcal{L}, \mathcal{O}_X).$$

Let us first assume $r > 1$. Since $\mathcal{F}$ is μ_H-stable, we have necessary conditions for the existence of a line subbundle of $\mathcal{F}^{[n]}$:

$$l.H < \frac{f.H}{r} \text{ and } l.H \leq 0. \tag{14}$$

The first inequality is due to the stability of $\mathcal{F}$ and the second comes from the fact that if a line bundle has a section, its first Chern class has non-negative intersection with any ample class H. If $\mathcal{L}_{X^{[n]}} \subset \mathcal{F}^{[n]}$ is destabilising, by Corollary 3.14 we must have

$$l.H \geq \frac{f.H}{nr}.$$

But this is certainly a contradiction to (14).

If $r = 1$, we can proceed as above but additionally have to consider the special case $\mathcal{L} \simeq \mathcal{F}$, i.e. $l.H = f.H$. The destabilising condition together with $l.H \leq 0$ immediately yields $l.H = 0$. But now $\mathrm{Hom}_X(\mathcal{L}, \mathcal{O}_X)$ can only be nontrivial if $\mathcal{L} \simeq \mathcal{O}_X$. □

As a direct generalisation of Lemma 4.1 we have:

Lemma 4.10. *For all locally free sheaves $\mathcal{G}$ and $\mathcal{H}$ on $X \times X^{[n-1]}$ and all $a \in \mathbb{Z}$ we have:*

$$\mathrm{Hom}_{X^{[n-1,n]}}(\sigma_n^\star \mathcal{G} \otimes \mathcal{O}(aD_n), \sigma_n^\star \mathcal{H}) \quad \subseteq \quad \mathrm{Hom}_{X \times X^{[n-1]}}(\mathcal{G}, \mathcal{H})$$

Now we consider arbitrary line subbundles $\mathcal{L}_{X^{[n]}} \otimes \mathcal{O}(a\delta_n)$, $\mathcal{L} \in \mathrm{Pic}\, X$, $a \in \mathbb{Z}$, and show that we can reduce to the case of Proposition 4.9:

Lemma 4.11. *Let $\mathcal{L}_{X^{[n]}} \otimes \mathcal{O}(a\delta_n)$ be a line bundle on $X^{[n]}$, Then for any locally free sheaf $\mathcal{F}$ on X we have*

$$\mathrm{Hom}_{X^{[n]}}(\mathcal{L}_{X^{[n]}} \otimes \mathcal{O}(a\delta_n), \mathcal{F}^{[n]}) \quad \subseteq \quad \mathrm{Hom}_{X^{[n]}}(\mathcal{L}_{X^{[n]}}, \mathcal{F}^{[n]}).$$

Proof. We use Remark 3.2, adjunction, the recursive formulas in Corollary 1.19 and

Lemma 1.20, Remark 1.21, Lemma 4.10 above and, finally, the Künneth formula:

$$\begin{aligned}
& \operatorname{Hom}_{X^{[n]}}(\mathcal{L}_{X^{[n]}} \otimes \mathcal{O}(a\delta_n), \mathcal{F}^{[n]}) \\
\cong\; & \operatorname{Hom}_{X^{[n]}}(\mathcal{L}_{X^{[n]}} \otimes \mathcal{O}(a\delta_n), \psi_{n\star}\sigma_n^\star q_{n-1}^\star \mathcal{F}) \\
\cong\; & \operatorname{Hom}_{X^{[n-1,n]}}(\psi_n^\star(\mathcal{L}_{X^{[n]}} \otimes \mathcal{O}(a\delta_n)), \sigma_n^\star q_{n-1}^\star \mathcal{F}) \\
\cong\; & \operatorname{Hom}_{X^{[n-1,n]}}\big(\sigma_n^\star\big(p_{n-1}^\star(\mathcal{L}_{X^{[n-1]}} \otimes \mathcal{O}(a\delta_{n-1})) \otimes q_{n-1}^\star \mathcal{L}\big) \otimes \mathcal{O}(aD_n), \sigma_n^\star q_{n-1}^\star \mathcal{F}\big) \\
\subseteq\; & \operatorname{Hom}_{X \times X^{[n-1]}}(p_{n-1}^\star(\mathcal{L}_{X^{[n-1]}} \otimes \mathcal{O}(a\delta_{n-1})) \otimes q_{n-1}^\star \mathcal{L}, q_{n-1}^\star \mathcal{F}) \\
\cong\; & \operatorname{Hom}_X(\mathcal{L}, \mathcal{F}) \otimes \operatorname{Hom}_{X^{[n-1]}}(\mathcal{L}_{X^{[n-1]}} \otimes \mathcal{O}(a\delta_{n-1}), \mathcal{O}_{X^{[n-1]}}).
\end{aligned}$$

Now we can use induction to conclude. The initial step for $n = 2$ is settled by Lemma 4.1. □

We are ready to prove the first main result of this section.

Proposition 4.12. *Let $\mathcal{F}$ be a torsion-free μ_H-stable sheaf on X. Assume that its reflexive hull $\mathcal{F}^{\vee\vee} \not\cong \mathcal{O}_X$. Then $\mathcal{F}^{[n]}$ does not contain μ_{H_N}-destabilising subsheaves of rank one for all $N \gg 0$.*

Proof. If $\mathcal{F}$ is locally free, we can simply apply Proposition 4.9 and Lemma 4.11 above. If $\mathcal{F}$ is not locally free, we proceed as usual in order to reduce to the locally free case:

Let $\mathcal{E} := \mathcal{F}^{\vee\vee}$ be the reflexive hull of $\mathcal{F}$. It has the same rank and first Chern class and is a locally free μ_H-stable sheaf. Thus we get an injection of $\mathcal{F}^{[n]}$ into the locally free tautological sheaf $\mathcal{E}^{[n]}$ which again has the same rank and first Chern class. Now we can apply the lemmata above. Note that every destabilising subsheaf of $\mathcal{F}^{[n]}$ also destabilises $\mathcal{E}^{[n]}$. □

Since the tautological sheaf on $X^{[3]}$ associated with a rank one sheaf has rank three, the above proposition is enough to show that these sheaves are stable (except $\mathcal{O}_X^{[3]}$, of course).

Theorem 4.13. *Let $\mathcal{F}$ be a torsion-free rank one sheaf on X satisfying $\det \mathcal{F} \not\cong \mathcal{O}_X$. Then for all sufficiently large N the associated rank three sheaf $\mathcal{F}^{[3]}$ on $X^{[3]}$ is μ_{H_N}-stable.*

Proof. As usual we can reduce to the case that $\mathcal{F}$ is locally free. We have seen that $\mathcal{F}^{[3]}$ cannot contain destabilising subsheaves of rank one. But any destabilising subsheaf of rank two yields a rank one destabilising subsheaf of the dual sheaf. Using Lemma 3.4 we are done. □

4.4 The Case of the Trivial Line Bundle

In the preceding sections we explicitly excluded the case where $\mathcal{F}$ is isomorphic to the trivial line bundle. The proof of Corollary 4.3 fails and, in fact, we have the following result:

Proposition 4.14. *The rank n tautological vector bundle $\mathcal{O}_X^{[n]}$ associated with the trivial line bundle $\mathcal{O}_X$ is not μ_{H_N}-stable for sufficiently large N.*

Proof. By [Scal09b, Cor. 4.5] we have $\mathrm{H}^0(\mathcal{O}_X^{[n]}) \cong \mathbb{C}$. Thus the structure sheaf $\mathcal{O}_{X^{[n]}}$ is a line subbundle of $\mathcal{O}_X^{[n]}$. We compare the slopes in order to show that $\mathcal{O}_{X^{[n]}}$ is destabilising. Since $c_1(\mathcal{O}_{X^{[n]}}) = 0$, we surely have

$$\mu_{H_N}(\mathcal{O}_{X^{[n]}}) = 0.$$

And by 3.5 we have $c_1(\mathcal{O}_X^{[n]}) = -\delta_n$. But $2\delta_n$ is effective and for N sufficiently large H_N is ample. Hence

$$0 > -\delta_n H_N^{2n-1} = n \cdot \mu_{H_N}(\mathcal{O}_X^{[n]}).$$

□

5 Stability of Tautological Sheaves: Abelian Surfaces

In this chapter we try to transfer the results of Chapter 4 to the case of abelian surfaces A. In this case the structure of $\mathrm{NS}(A^n)$ is much more complicated. We prove certain stability results in the cases $n = 2$ and $n = 1$. Of high interest is the restriction of tautological sheaves to the associated generealised Kummer varieties for which we also prove stability in some cases.

5.1 Geometric considerations: $n = 2$

As explained already at the end of Section 1.3, for abelian surfaces A the Néron−Severi group $\mathrm{NS}(A \times A)$ is not isomorphic to $(\mathrm{NS}\, A)^{\boxplus 2}$. Thus we cannot simply apply the results of the preceding chapter to this case. In order to prove the stability of tautological sheaves nevertheless, we will restrict to the case of principally polarised abelian surfaces (p.p.a.s.) (A, H) of Picard rank one. We begin with a technical lemma:

Lemma 5.1. *Let A and A' be complex tori. Then we have an isomorphism of abelian groups*

$$\mathrm{NS}(A \times A') \cong \mathrm{NS}(A) \oplus \mathrm{Hom}(A', \widehat{A}) \oplus \mathrm{NS}(A'),$$

where $\widehat{A}$ denotes the dual torus.

Proof. The proof of this lemma was pointed out to me by H. Ohashi. The Künneth formula yields a decomposition of $\mathrm{NS}(A \times A')$ into direct summands, two of which are naturally isomorphic to $\mathrm{NS}(A)$ and $\mathrm{NS}(A')$, respectively. The remaining summand can be written as

$$\big((\mathrm{H}^{1,0}(A) \otimes \mathrm{H}^{0,1}(A')) \oplus (\mathrm{H}^{0,1}(A) \otimes \mathrm{H}^{1,0}(A'))\big) \cap \big(\mathrm{H}^1(A, \mathbb{Z}) \otimes \mathrm{H}^1(A', \mathbb{Z})\big). \qquad (15)$$

We can interpret $\mathrm{H}^{1,0}(A) \otimes \mathrm{H}^{0,1}(A')$ as $\mathrm{Hom}(\mathrm{H}^{0,1}(\widehat{A}), \mathrm{H}^{0,1}(A'))$ and so we see that (15) is just the set of morphisms of integral Hodge structures

$$\mathrm{H}^1(\widehat{A}, \mathbb{Z}) \to \mathrm{H}^1(A', \mathbb{Z}). \qquad \square$$

Denote by $s\colon A \times A \to A$ the group law. An important role will play the class $H_M := s^\star H - H^{\boxplus 2}$, which is called the *Mumford class* associated with H. By Lemma 1.17 it is linearly independent from the summand $(\mathrm{NS}\, A)^{\boxplus 2}$ inside $\mathrm{NS}(A \times A)$. We will restrict to a certain (large) set of p.p.a.s.:

$$A \text{ is a p.p.a.s. such that} \qquad \begin{pmatrix} \mathrm{NS}\, A \cong \mathbb{Z}H \\ \text{and} \\ \mathrm{NS}(A \times A) \cong \mathrm{NS}\, A^{\boxplus 2} \oplus \mathbb{Z}H_M. \end{pmatrix} \quad (\star)$$

Lemma 5.2. *The class of abelian surfaces satisfying condition* ($\star$) *is the complement of a countable union of analytic subsets in the moduli space $\mathcal{A}_2$ of principally polarised abelian surfaces.*

Proof. It is well known that the set of abelian surfaces of Picard rank one is very general in $\mathcal{A}_2$ (cf. [BL92, Exerc. 1a), p. 244]). So we can assume $\mathrm{NS}\, A \cong \mathbb{Z}H$, where H is a principal polarisation. For the second assertion note that we always have $\mathrm{NS}\, A^{\boxplus 2} \subset \mathrm{NS}(A \times A)$ and there is at least one additional summand containing the class H_M. In order to show that for a very general abelian surface no more summands show up, we use Theorem 9.1 in [BL92] (or Theorem 4.7.1 in [Ara]) — both stating that a very general A satisfies $\mathrm{Hom}(A, A) \cong \mathbb{Z}$ — and Lemma 5.1 above. Finally, let us show that the class H_M is primitive. By Lemma 5.1 it is certainly enough to prove that H_M corresponds to the identity in $\mathrm{Hom}(A, A)$. To the class H we can associate the homomorphism

$$\varphi_H \colon A \to \widehat{A}, \quad x \mapsto t_x^\star \mathcal{O}(H) \otimes \mathcal{O}(-H),$$

which is an isomorphism since H is a principal polarisation and it yields an identification $\mathrm{Hom}(A, A) \cong \mathrm{Hom}(A, \widehat{A})$. Furthermore, on $A \times \widehat{A}$ we have the Poincaré line bundle $\mathcal{P}$. Altogether we can describe the inclusion $\mathrm{Hom}(A, A) \hookrightarrow \mathrm{NS}(A \times A)$ as follows:

$$\begin{array}{ccccc} \mathrm{Hom}(A, A) & \to & \mathrm{Hom}(A, \widehat{A}) & \to & \mathrm{NS}(A \times A) \\ f & \mapsto & \varphi_H \circ f, & & \\ & & \varphi & \mapsto & c_1((\mathrm{id}_A \times \varphi)^\star \mathcal{P}). \end{array}$$

Finally, equation (9.8) in [Huy06, Chapt. 9] says that

$$(\mathrm{id}_A \times \varphi_H)^\star \mathcal{P} \simeq s^\star \mathcal{O}(H) \otimes \mathcal{O}(-H)^{\boxtimes 2}.$$

□

Remark 5.3. By the considerations on Page 198 in [Huy06] we see that the first Chern class of $(\mathrm{id}_A \times \varphi_H)^\star \mathcal{P}$ is contained in the Künneth summand $H^1(A, \mathbb{Z})^{\boxtimes 2}$. Thus the identity

$$s^\star H = H^{\boxplus 2} + H_M$$

is exactly the Künneth decomposition.

If H is not a principal polarisation, the homomorphism φ_H is no longer an isomorphism. Thus it is not clear if the class H_M is primitive.

Corollary 5.4. *If A satisfies* ($\star$), *we have*

$$\mathrm{NS}\, A^{[2]} \cong \mathbb{Z}H_{A^{[2]}} \oplus \mathbb{Z}H_m \oplus \mathbb{Z}\delta.$$

Proof. This follows easily from the assumption ($\star$) and the fact that $\widetilde{A \times A} \xrightarrow{\psi} A^{[2]}$ is the $\mathfrak{S}_2$-quotient. □

We continue by deriving intersection numbers and slopes for the case $n = 2$.

Lemma 5.5. *We have the following identities in* $\mathrm{H}^8(A \times A, \mathbb{Z}) \cong \mathbb{Z}$*:*

$$\begin{aligned} s^\star H^2 \cdot (H \otimes H) &= 4, \qquad (16)\\ s^\star H^2 \cdot (1 \otimes H^2) = s^\star H^2 \cdot (H^2 \otimes 1) &= 4 \quad \textit{and} \\ s^\star H \cdot (H \otimes H^2) = s^\star H \cdot (H^2 \otimes H) &= 4. \end{aligned}$$

Proof. The first equality is true for all p.p.a.s. A if and only if it is true for one. Thus we may choose A to be the product of two elliptic curves E and E' and we represent H by $E \times \{0\} + \{0\} \times E'$. We shall prove the lemma by replacing this representation of the class H by appropiate translates such that the intersection (16) becomes transversal and then calculate the set-theoretic intersection. Thus let x_1, x_2 and y_1, y_2 be points on E and E', respectively. For all but a finite number of choices of these four points, the following intersection in $A \times A = (E \times E') \times (E \times E')$ is transversal:

$$\begin{aligned} & s^{-1}(H + (x_1, y_1)) \cap s^{-1}(H + (x_2, y_2)) \cap \pi_1^{-1} H \cap \pi_2^{-1} H \\ = & \ \big\{(0, y_2, x_1, 0), (0, y_1, x_2, 0), (x_1, 0, 0, y_2), (x_2, 0, 0, y_1)\big\}. \end{aligned}$$

The other equalities can be proven in the same way. □

Corollary 5.6. *We have*

$$\begin{aligned} H_M(H^2 \otimes H) = H_M(H \otimes H^2) &= 0 \quad \textit{and} \qquad (17)\\ (H_M)^2 \cdot H \otimes H &= -4. \end{aligned}$$

Proof. The first equality follows directly from the lemma above. Furthermore, we have

$$(H_M)^2 = s^\star H^2 - 2 s^\star H \cdot H^{\boxplus 2} + H^2 \otimes 1 + 1 \otimes H^2 + 2H \otimes H.$$

Intersecting with $H \otimes H$ yields

$$(H_M)^2 \cdot H \otimes H = 4 - 2 \cdot 2 \cdot 4 + 0 + 0 + 2 \cdot 4 = -4. \qquad \square$$

In the case of regular surfaces we used the polarisation $H_N := N H_{X^{[2]}} - \delta$. The following lemma indicates that this polarisation is not the ideal choice in the abelian surface case. Note that in analogy to the definition of the Mumford class we defined

$$H_m := m^\star H - H_{A^{[2]}}.$$

Lemma 5.7. *We have the following expansion:*

$$H_m \cdot H_N^3 = 0 + O(N^2).$$

Proof. We pullback along the double cover $\psi \colon \widetilde{X \times X} \to X^{[2]}$. Note that $\psi^\star H_{X^{[2]}} = \sigma^\star H^{\boxplus 2}$ and $\psi^\star H_m = \sigma^\star H_M$. We have

$$\begin{aligned} H_m \cdot H_N^3 &= \tfrac{1}{2} \sigma^\star H_M \cdot (\sigma^\star H_N)^3 = \tfrac{1}{2} \sigma^\star (H_M \cdot (H^{\boxplus 2})^3) + O(N^2) \\ &= \tfrac{1}{2} H_M (3H^2 \otimes H + 3H \otimes H^2) + O(N^2). \end{aligned}$$

Now we use (17) in Corollary 5.6 and we are done. $\square$

Thus the leading term in the expansion of the slope of a line bundle with first Chern class H_m is zero.

In Lemma 3.15 we defined the polarisation $H_N^m := NH_{X^{[2]}} - \delta + Nm^\star H$. With respect to this polarisation the slope of line bundles with first Chern class $m^\star H$ does not vanish:

Lemma 5.8. *We have the following expansions:*

$$\begin{aligned}(H_N^m)^3 H_{A^{[2]}} &= 72N^3 + O(N^2), \\ (H_N^m)^3 H_m &= -36N^3 + O(N^2) \quad \textit{and} \\ (H_N^m)^3 m^\star H &= 36N^3 + O(N^2).\end{aligned}$$

Proof. First observe that by definition

$$(H_N^m)^3 m^\star H = (H_N^m)^3 H_{A^{[2]}} + (H_N^m)^3 H_m.$$

Therefore, we will only prove the first two equalities. Note that we have $H^3 \otimes 1 = 1 \otimes H^3 = 0 = s^\star H^3$. We write down the expansion of $\psi^\star(H_N^m)^3$:

$$\begin{aligned}\psi^\star(H_N^m)^3 &= N^3\psi^\star(H_{X^{[2]}} + m^\star H)^3 + O(N^2) \\ &= N^3\sigma^\star\big(3(H^2\otimes H + H\otimes H^2) + 3s^\star H(H^2\otimes 1 + 2H\otimes H + 1\otimes H^2) \\ &\quad +3s^\star H^2(H\otimes 1 + 1\otimes H)\big) + O(N^2). \end{aligned} \tag{18}$$

Using Lemma 5.5, we have

$$\begin{aligned}\psi^\star((H_N^m)^3 H_{A^{[2]}}) &= \psi^\star(H_N^m)^3\sigma^\star H^{\boxplus 2} \\ &= N^3\big(6H^2\otimes H^2 + 18s^\star H(H^2\otimes H) + 6s^\star H^2(H^2\otimes 1) \\ &\quad + 6s^\star H^2(H\otimes H)\big) + O(N^2) \\ &= N^3(6\cdot 4 + 18\cdot 4 + 6\cdot 4 + 6\cdot 4) + O(N^2) \\ &= 144N^3 + O(N^2).\end{aligned}$$

For the second equality we use Corollary 5.6: We do not have to consider the term $3(H^2\otimes H + H\otimes H^2)$ in the expansion (18). Thus we have

$$\begin{aligned}\psi^\star((H_N^m)^3 H_m) &= \psi^\star(H_N^m)^3\sigma^\star(s^\star H - H^{\boxplus 2}) \\ &= N^3\big(3s^\star H^2(H^2\otimes 1 + 2H\otimes H + 1\otimes H^2) - 18s^\star H(H\otimes H^2) - 6s^\star H^2(H\otimes H) \\ &\quad - 6s^\star H^2(H^2\otimes 1)\big) + O(N^2) \\ &= N^3(3\cdot(4 + 2\cdot 4 + 4) - 18\cdot 4 - 6\cdot 4 - 6\cdot 4 + O(N^2) \\ &= -72N^3 + O(N^2).\end{aligned}$$

$\square$

5.2 Geometric considerations: $n = 3$

Now we turn to the case $n = 3$ which is not essentially different from the case $n = 2$ but the calculations are somewhat more difficult. We still assume that (A, H) is an abelian surface satisfying $(\star)$. We denote the projections $A^3 \to A$ by π_i, $i = 1, 2, 3$ and the projections $A^3 \to A^2$ by p_{jk}, $1 \leq j < k \leq 3$. If A satisfies $(\star)$, a similar analysis as in Lemma 5.1 yields

$$\operatorname{NS} A^3 \cong \bigoplus_{i=1}^{3} \mathbb{Z}\pi_i^\star H \oplus \bigoplus_{1\leq j<k\leq 3} \mathbb{Z} p_{jk}^\star H \tag{19}$$

Note that no more complicated effects occur since the Néron−Severi group is a subgroup of the second (!) cohomology. An important class in $\operatorname{NS} A^3$ is H_M. We need the following lemma to write this class according to decomposition (19).

Lemma 5.9. *We have*

$$s_3^\star H = -\sum_{i=1}^{3} \pi_i^\star H + \sum_{1\leq j<k\leq 3} p_{jk}^\star s^\star H.$$

Proof. Certainly $s_3^\star H$ is $\mathfrak{S}_3$-invariant. Hence we can write

$$s_3^\star H = a\sum_{i=1}^{3} \pi_i^\star H + b \sum_{1\leq j<k\leq 3} p_{jk}^\star s^\star H, \quad a, b \in \mathbb{Z}.$$

We will intersect with different curve classes to determine a and b. Fix points x_0 and y_0 in A and intersect with the class $l_1 := \{x_0\} \times \{y_0\} \times H$. We have $s_3^\star H \cdot l_1 = \pi_3^\star H \cdot l_1 = p_{13}^\star s^\star H \cdot l_1 = p_{23}^\star s^\star H \cdot l_1 = H^2$ and all other intersections vanish. This yields $1 = a + 2b$. Next, we intersect with the class $l_2 := \{x_0\} \times \Delta_\star H$. We have $\pi_2^\star H \cdot l_2 = \pi_2^\star H \cdot l_2 = p_{12}^\star s^\star H \cdot l_2 = p_{13}^\star s^\star H \cdot l_2 = H^2$ but $s_3^\star H \cdot l_2$ consists of triples (x, y, z) such that $x = x_0$, $y = z \in H$ and $x_0 + 2z \in H$. For a general x_0 this gives H^2 multiplied by the number of two-torsion points, which is 16. Furthermore, we get the same number for $p_{23}^\star m^\star H \cdot l_2$ and the remaining term vanishes. Altogether we get the following system of linear equations:

$$\begin{aligned} 1 &= a + 2b \\ 16 &= 2a + 18b, \end{aligned}$$

which implies $a = -1$, $b = 1$. □

Corollary 5.10. *We have*

$$H_{M_3} = \sum_{1\leq j<k\leq 3} p_{jk}^\star H_M.$$

Proof. We just plug in the definition of H_M from (6) at the end of Section 1.4. We see that $s_3^\star = \sum_{i=1}^{3} \pi_i^\star H + \sum_{1\leq j<k\leq 3} p_{jk}^\star H_M$. □

From (19) and Corollary 5.10 we deduce

$$\mathrm{NS}\, A^{[3]} \;\cong\; \mathbb{Z}H_{A^{[3]}} \oplus \mathbb{Z}H_{m_3} \oplus \mathbb{Z}\delta_3.$$

Recall that we defined our polarisation H_N^m as follows:

$$H_N^m := N(H_{A^{[3]}} + m_3^\star H) - \delta_3.$$

In order to calculate slopes of sheaves on $A^{[3]}$, we need to have an expansion of certain intersection products in terms of N.

Lemma 5.11. *We have:*

$$(H_N^m)^5 \cdot H_{A^{[3]}} \;=\; 3 \cdot (H_N^m)^5 \cdot m_3^\star H + O(N^4). \tag{20}$$

Proof. We have

$$(H_N^m)^5 \;=\; N^5\big(H^5_{A^{[3]}} + 5H^4_{A^{[3]}} m_3^\star H + 10 H^3_{A^{[3]}}(m_3^\star H)^2\big) + O(N^4)$$

and we therefore need to compute all terms of the form $H^i_{A^{[3]}}(m_3^\star H)^j$ with $i + j = 6$, $j \leq 2$. (Note that $(m_3^\star H)^j$ vanishes for $j > 2$.) All the divisors involved are actually pullbacks along the Hilbert−Chow morphism ρ and we can therefore compute them on S^3A or, even more easily, their pullbacks on A^3. Since we are not interested in the exact value but only want to compare both sides of (20), we do not care about scaling factors like the factor 6 when we pull back to A^3. To make computations easier we define the class

$$H_s := m_3^\star H + H_{A^{[3]}}.$$

The images of the classes $H_{A^{[3]}}$ and H_s on A^3 are $H^{\boxplus 3}$ by Lemma 1.20 and $\sum_{1\leq j<k\leq 3} p_{jk}^\star s^\star H$ by Lemma 5.9, respectively. For degree reasons many terms vanish. We use the fact that

$$s^\star H = H^{\boxplus 2} + H_M$$

is the decomposition into Künneth factors (cf. Remark 5.3). We use Lemma 5.5 to compute the remaining terms:

$$\begin{aligned}
H^6_{A^{[3]}} &= \binom{6}{2,2,2} \cdot \pi_1^\star H^2 \cdot \pi_2^\star H^2 \cdot \pi_3^\star H^2 = 90(H^2)^3,\\
H^5_{A^{[3]}} H_s &= 3 \cdot 2 \cdot \binom{5}{2,2,1} \cdot (p_{12}^\star s^\star H \cdot \pi_1^\star H \cdot \pi_2^\star H^2 \cdot \pi_3^\star H^2)\\
&= 3 \cdot 2 \cdot 30 (H^2)^3 = 180 (H^2)^3,
\end{aligned}$$

$$
\begin{aligned}
H_{A^{[3]}}^4 H_s^2 &= 3\cdot (H^{\boxplus 3})^4 \cdot p_{12}^\star s^\star H^2 + 6\cdot (H^{\boxplus 3})^4\cdot p_{12}^\star s^\star H \cdot p_{23}^\star s^\star H \\
&= 3\cdot \Big(2\cdot \binom{4}{2,0,2}\cdot \pi_1^\star H^2\cdot \pi_3^\star H^2 + \binom{4}{1,1,2}\cdot \pi_1^\star H\cdot \pi_2^\star H\cdot \pi_3^\star H^2\Big)\cdot p_{12}^\star s^\star H^2 \\
&\quad + 6\cdot \Big(2\cdot \binom{4}{1,1,2}\cdot \pi_1^\star H\cdot \pi_2^\star H\cdot \pi_3^\star H^2 + \binom{4}{1,2,1}\cdot \pi_1^\star H\cdot \pi_2^\star H^2\cdot \pi_3^\star H \\
&\quad + \binom{4}{2,0,2}\cdot \pi_1^\star H^2\cdot \pi_3^\star H^2\Big)\cdot p_{12}^\star s^\star H\cdot p_{23}^\star s^\star H \\
&= 3\cdot \big(2\cdot 6\cdot (H^2)^3 + 12\cdot (H^2)^3\big) + 6\cdot \big(2\cdot 12\cdot (H^2)^3 + 12\cdot (H^2)^3 + 6\cdot (H^2)^3\big) \\
&= 324(H^2)^3.
\end{aligned}
$$

Now we resubstitute $m_3^\star H = H_s - H_{A^{[3]}}$. We have

$$
\begin{aligned}
H_{A^{[3]}}^5 m_3^\star H &= (180-90)(H^2)^3 = 90(H^2)^3 \\
H_{A^{[3]}}^4 (m_3^\star H)^2 &= (324 - 2\cdot 180 + 90)(H^2)^3 = 54(H^2)^3.
\end{aligned}
$$

Altogether we have

$$
\begin{aligned}
(H_N^m)^5\cdot H_{A^{[3]}} &= N^5\big(90+5\cdot 90+10\cdot 54\big)\cdot (H^2)^3 + O(N^4) = 1080\cdot (H^2)^3 + O(N^4) \\
&\quad\text{and} \\
(H_N^m)^5\cdot m_3^\star H &= N^5\big(90+5\cdot 54\big)\cdot (H^2)^3 + O(N^4) = 360\cdot (H^2)^3 + O(N^4).
\end{aligned}
$$

□

Remark 5.12. The exact values in equation (20) in Lemma 5.11 are, of course, of no interest. Much more important is the fact that the degree of H and $3m_3^\star H$ with respect to H_N^m is the same.

5.3 The Case $n = 2$

Now we will apply the computations of Section 5.1 and obtain similar stability results as in the case of regular surfaces in Section 4.2. We proceed similarly and will first exclude destabilising line subbundles. Note that for a p.p.a.s. satisfying $(\star)$ we can write every line bundle $\mathcal{L}$ on $\widetilde{A\times A}$ as

$$\mathcal{L} \simeq r_1^\star \mathcal{M}_1 \otimes r_2^\star \mathcal{M}_2 \otimes \sigma^\star \mathcal{O}(bs^\star H)\otimes \mathcal{O}(cD)$$

with $\mathcal{M}_i \in \operatorname{Pic} A$ and $b, c\in \mathbb{Z}$.

Proposition 5.13. *Let A be a p.p.a.s. satisfying $(\star)$ and let $\mathcal{F}$ be a μ_H-stable vector bundle on A of rank r and first Chern class $c_1(\mathcal{F}) = fH$, $f\in\mathbb{Z}$. Then $r_1^\star\mathcal{F}$ does not contain any line bundle $\mathcal{L}\simeq r_1^\star \mathcal{M}_1 \otimes r_2^\star \mathcal{M}_2 \otimes \sigma^\star \mathcal{O}(bs^\star H)\otimes \mathcal{O}(cD)$ with $c_1(\mathcal{M}_i) = a_iH$, $a_i, b, c\in\mathbb{Z}$ satisfying the* tautological destabilising condition

$$a_1 + a_2 + b \geq \frac{f}{r},$$

but in the case $\mathcal{F} \simeq \mathcal{M}_1$, $\mathcal{M}_2 \simeq \mathcal{O}_A$, $b = c = 0$.

Remark 5.14. Note that the *tautological destabilising condition* does not contain c at all. This is in perfect analogy to Corollary 4.3.

Proof. We consider a line bundle $\mathcal{L} \simeq r_1^\star \mathcal{M}_1 \otimes r_2^\star \mathcal{M}_2 \otimes \sigma^\star \mathcal{O}(bs^\star H) \otimes \mathcal{O}(cD)$ with $c_1(\mathcal{M}_i) = a_i H$ and $a_i, b, c \in \mathbb{Z}$. Note that since $\sigma^\star \mathcal{O}(bs^\star H)$ comes from $A \times A$, we can assume — analogously to Lemma 4.1 — that $c = 0$. Thus it is enough to show that

$$\operatorname{Hom}_{A\times A}(\pi_1^\star \mathcal{M}_1 \otimes \pi_2^\star \mathcal{M}_2 \otimes s^\star \mathcal{O}(bH), \pi_1^\star \mathcal{F}) \tag{21}$$

vanishes. To prove this we use adjunction ($\pi_1^\star \dashv \pi_{1\star}$, $\pi_2^\star \dashv \pi_{2\star}$ and $s^\star \dashv s_\star$) to get the following three different representations of this vector space:

$$\begin{aligned}
(21) \;\cong\;& \operatorname{Hom}_A\left(\mathcal{M}_1, \mathcal{F} \otimes \pi_{1\star}(\pi_2^\star \mathcal{M}_2^\vee \otimes s^\star \mathcal{O}(-bH))\right) && (22)\\
\cong\;& \operatorname{Hom}_A\left(\mathcal{M}_2, \pi_{2\star}(\pi_1^\star(\mathcal{F} \otimes \mathcal{M}_1^\vee) \otimes s^\star \mathcal{O}(-bH))\right) &&\\
\cong\;& \operatorname{Hom}_A\left(\mathcal{O}(bH), s_\star(\pi_1^\star(\mathcal{F} \otimes \mathcal{M}_1^\vee) \otimes \pi_2^\star \mathcal{M}_2^\vee)\right). && (23)
\end{aligned}$$

According to these three representations we shall consider three cases.

a) $a_2 + b \geq 0$: The restriction of $\pi_2^\star \mathcal{M}_2^\vee \otimes s^\star \mathcal{O}(-bH)$ to a fibre $\pi_1^{-1}(x)$ is isomorphic to $\mathcal{M}_2^\vee \otimes t_x^\star \mathcal{O}(-bH)$, where $t_x \colon A \to A$ is the translation by x. This is a line bundle with first Chern class $-(a_2 + b)H$ on A. Thus if $a_2 + b > 0$, the space of global sections on the fibres is trivial and so is (22). If $a_2 + b = 0$ and $b \neq 0$, then $\mathrm{H}^0(\mathcal{M}_2^\vee \otimes t_x^\star \mathcal{O}(-bH))$ is zero outside a finite number of $x \in A$. (Note that since H is an ample class, we have $\#\{x \in A \mid t_x^\star \mathcal{O}(-bH) \simeq \mathcal{M}_2^\vee\} < \infty$.) Hence $\pi_{1\star}(\pi_2^\star \mathcal{M}_2^\vee \otimes s^\star \mathcal{O}(-bH))$ would have finite support but since it is torsion-free, it vanishes. The remaining case is $a_2 = b = 0$. Now $\mathrm{H}^0(\mathcal{M}_2^\vee \otimes t_x^\star \mathcal{O}(-bH)) = \mathrm{H}^0(\mathcal{M}_2^\vee)$ vanishes but in the case $\mathcal{M}_2 \simeq \mathcal{O}_A$. Furthermore, (22) equals $\operatorname{Hom}(\mathcal{M}_1, \mathcal{F})$. The destabilising condition yields $a_1 \geq \frac{f}{r}$ which implies $\operatorname{Hom}(\mathcal{M}_1, \mathcal{F}) = 0$ but in the case $\mathcal{F} \simeq \mathcal{M}_1$.

b) $a_2 < 0$: Similar to above we consider the restriction of $\pi_1^\star(\mathcal{F} \otimes \mathcal{M}_1^\vee) \otimes s^\star \mathcal{O}(-bH)$ to a fibre $\pi_2^{-1}(x)$. Taking global sections, this yields

$$\mathrm{H}^0(\mathcal{F} \otimes \mathcal{M}_1^\vee \otimes t_x^\star \mathcal{O}(-bH)) \cong \operatorname{Hom}(\mathcal{M}_1 \otimes t_x^\star \mathcal{O}(bH), \mathcal{F}),$$

which, by the stability of $\mathcal{F}$, has to vanish since the destabilising condition implies $a_1 + b > \frac{f}{r}$.

c) $b < 0$: Analogously to b) we now use (23). This time the destabilising condition yields $a_1 + a_2 > \frac{f}{r}$. □

From Lemma 5.8 and Proposition 5.13 above we can deduce:

Corollary 5.15. *Assume* $\mathcal{F} \not\simeq \mathcal{O}_A$. *Then for* $N \gg 0$ *there are no* $\mu_{H_N^m}$*-destabilising line subbundles in* $\mathcal{F}^{[2]}$.

Proof. Let $\mathcal{L}'$ be a destabilising line subbundle of $\mathcal{F}^{[2]}$. We write its pullback $\mathcal{L} := \psi^\star\mathcal{L}'$ as $\mathcal{L} = \sigma^\star(\mathcal{M}^{\boxtimes 2} \otimes \mathcal{O}(bs^\star H)) \otimes \mathcal{O}(cD)$ with $c_1(\mathcal{M}) = aH$, $a \in \mathbb{Z}$. As usual, by adjunction we get a homomorphism $\mathcal{L} \to r_1^\star\mathcal{F}$. By Lemma 5.8 the destabilising condition on $\mathcal{L}$ yields

$$2a + b \geq \frac{f}{r},$$

which implies that $\mathcal{L}$ satisfies the *tautological destabilising condition* from Proposition 5.13. □

As in Section 4.2 this result on destabilising line subbundles suffices to prove the stability of rank two locally free tautological sheaves. We use the same argument as in the proofs of Theorem 4.6 and Proposition 4.12 to generalise to the torsion-free case.

Theorem 5.16. *Let $\mathcal{F}$ be a rank one torsion-free sheaf on A satisfying $det\mathcal{F} \not\simeq \mathcal{O}_A$. Then the rank two tautological sheaf $\mathcal{F}^{[2]}$ is $\mu_{H_N^m}$-stable for sufficiently large N.*

Next, we prove the analogue of Theorem 4.8. Since the proof is almost the same, we only touch upon the crucial parts.

Theorem 5.17. *Let $\mathcal{F}$ be a rank two μ_H-stable sheaf on A and assume* $\det\mathcal{F} \not\simeq \mathcal{O}_A$. *Then for N sufficiently large $\mathcal{F}^{[2]}$ is a $\mu_{H_N^m}$-stable rank four sheaf on $A^{[2]}$.*

Proof. As above we may assume that $\mathcal{F}$ is locally free. We write $c_1(\mathcal{F}) = fH$, $f \in \mathbb{Z}$. Let $\mathcal{E}$ be the maximal destabilising sheaf of $\mathcal{F}^{[2]}$. Write $c_1(\mathcal{E}) = eH_{A^{[2]}} + gm^\star H + a\delta$, $e, g, a \in \mathbb{Z}$. We only consider the case that $\mathcal{E}$ is of rank two. We use the same notation as in the proof of Theorem 4.8:

If $\operatorname{rk}\ker\beta = 0$, we must have that

$$r_1^\star \det\mathcal{F} \otimes \psi^\star \det\mathcal{E}^\vee$$

has a section. From this we deduce that either $a < 0$ and $\det\mathcal{F} \simeq \mathcal{O}_A$ (which we excluded) or $a \leq 0$ and the class

$$(f - e)H \otimes 1 - 1 \otimes e - g\sigma^\star s^\star H$$

on $\widetilde{A \times A}$ is effective and nonzero. This time the evaluation against the polarisation $\psi^\star H_N^m$ gives a contradiction to the destabilising condition on $\mathcal{E}$.

If $\operatorname{rk}\ker\beta = 1$, we write $c_1(\operatorname{im}\beta) = l_1H \otimes 1 + 1 \otimes l_2 + h\sigma^\star s^\star H + bD$ with $l_1, l_2, h, b \in \mathbb{Z}$. The semistability of $\mathcal{E}$ yields

$$2e + g \leq 2(l_1 + l_2 + h)$$

and the destabilising condition on $\mathcal{E}$ implies

$$2e + g \geq f.$$

Thus we found a line bundle in $r_1^\star \mathcal{F}$ with

$$2(l_1 + l_2 + h) \geq f$$

contradicting Proposition 5.13. $\square$

5.4 The Case $n = 3$

In this section we prove the following result:

Theorem 5.18. *Let A be a p.p.a.s. satisfying $(\star)$ and let $\mathcal{F}$ be a rank one torsion-free sheaf on A satisfying $\det \mathcal{F} \not\simeq \mathcal{O}_A$. Then for all N sufficiently large $\mathcal{F}^{[3]}$ is $\mu_{H_N^m}$-stable.*

We denote the projection $A^3 \to A$ to the i-th factor by π_i and — in analogy to the case $n = 2$ — we begin by analysing line subbundles of the sheaf $\pi_1^\star \mathcal{F}$ on A^3.

Proposition 5.19. *Let $\mathcal{F}$ be a μ_H-stable vector bundle on A of rank r and first Chern class fH, $f \in \mathbb{Z}$. Then $\pi_1^\star \mathcal{F}$ does not contain any line subbundles of the form*

$$\mathcal{L} = \mathcal{M}^{\boxtimes 3} \otimes s_3^\star \mathcal{O}(bH),$$

$\mathcal{M} \in \operatorname{Pic} A$, $c_1(\mathcal{M}) = aH$, $a, b \in \mathbb{Z}$, *satisfying the* tautological destabilising condition

$$3a + b \geq \frac{f}{r},$$

but in the case $a = b = 0$, $\mathcal{M} \simeq \mathcal{F} \simeq \mathcal{O}_A$.

Proof. Again, we distinguish three cases:
a) $2a + b \geq 0$: We push forward along π_1:

$$\begin{aligned} \operatorname{Hom}(\mathcal{L}, \pi_1^\star \mathcal{F}) &\cong \mathrm{H}^0(\pi_1^\star \mathcal{F} \otimes \mathcal{L}^\vee) \\ &\cong \mathrm{H}^0(\mathcal{F} \otimes \mathcal{M}^\vee \otimes \pi_{1\star}(\underbrace{\pi_2^\star \mathcal{M}^\vee \otimes \pi_3^\star \mathcal{M}^\vee \otimes s_3^\star \mathcal{O}(-bH)}_{=:\mathcal{G}})). \end{aligned} \tag{24}$$

Restricting $\mathcal{G}$ to a fibre $\pi_1^{-1}(x)$, $x \in A$ yields (we identify $\pi_1^{-1}(x) = \{x\} \times A^2$, keep the notation for the projections π_i, $i = 2, 3$ but denote with s the multiplication of the second two factors):

$$\pi_2^\star \mathcal{M}^\vee \otimes \pi_3^\star \mathcal{M}^\vee \otimes s^\star t_x^\star \mathcal{O}(-bH).$$

The class of this line bundle on A^2 has degree $-(2a + b)$ with respect to the polarisation $H^{\boxplus 2} + s^\star H$ (cf. Lemma 5.8). Thus we get a contradiction but in the case $a = b = 0$ and $\mathcal{M} \simeq \mathcal{O}_A$. In this case we have (24) $\cong \operatorname{Hom}(\mathcal{O}_A, \mathcal{F})$, which vanishes by the stability of $\mathcal{F}$ but in the case $\mathcal{F} \simeq \mathcal{O}_A$.

b) $a < 0$: we push forward along π_2:

$$\begin{aligned} \mathrm{Hom}(\mathcal{L}, \pi_1^\star \mathcal{F}) &\cong \mathrm{H}^0(\pi_1^\star \mathcal{F} \otimes \mathcal{L}^\vee) \\ &\cong \mathrm{H}^0(\mathcal{M}^\vee \otimes \pi_{2\star}(\underbrace{\pi_1^\star(\mathcal{F} \otimes \mathcal{M}^\vee) \otimes \pi_3^\star \mathcal{M}^\vee \otimes s^\star \mathcal{O}(-bH)}_{=:\mathcal{H}})). \end{aligned}$$

Again, we restrict $\mathcal{H}$ to a fibre $\pi_2^{-1}(x)$, $x \in A$ and then take global sections:

$$\begin{aligned} \mathrm{H}^0(\mathcal{H}|_{\pi_2^{-1}(x)}) &\cong \mathrm{H}^0(\pi_1^\star(\mathcal{F} \otimes \mathcal{M}^\vee) \otimes \pi_3^\star \mathcal{M}^\vee \otimes s^\star t_x^\star \mathcal{O}(-bH)) \\ &\cong \mathrm{Hom}(\underbrace{\pi_1^\star \mathcal{M} \otimes \pi_3^\star \mathcal{M} \otimes s^\star t_x^\star \mathcal{O}(bH)}_{=:\mathcal{L}'}, \pi_1^\star \mathcal{F}). \end{aligned}$$

But this leads to a contradiction to Proposition 5.13: The first Chern class of $\mathcal{L}'$ is $aH^{\boxplus 2} + bs^\star H$ and by assumption we have

$$\frac{f}{r} \leq 3a + b < 2a + b.$$

c) $b < 0$: We push forward along s_3:

$$\begin{aligned} \mathrm{Hom}(\mathcal{L}, \pi_1^\star \mathcal{F}) &\cong \mathrm{H}^0(\pi_1^\star \mathcal{F} \otimes \mathcal{L}^\vee) \\ &\cong \mathrm{H}^0(\mathcal{O}(-bH) \otimes s_{3\star}(\underbrace{\pi_1^\star(\mathcal{F} \otimes \mathcal{M}^\vee) \otimes \pi_2^\star \mathcal{M}^\vee \otimes \pi_3^\star \mathcal{M}^\vee}_{=:\mathcal{E}})). \end{aligned}$$

The fibre of s_3 over a point $x \in A$ can be identified with A^2 as follows:

$$A^2 \to s_3^{-1}(x), \quad (y, z) \mapsto (y, z, x - (y + z)).$$

Under this identification π_1 and π_2 remain the same and π_3 is replaced by $t_x \circ \iota \circ s$, where $\iota \colon A \to A$ is the inverse. Thus we see that the restriction of $\mathcal{E}$ to $s_3^{-1}(x)$ is isomorphic to

$$\pi_1^\star \mathcal{F} \otimes \underbrace{\pi_1^\star \mathcal{M}^\vee \otimes \pi_2^\star \mathcal{M}^\vee \otimes s^\star \iota^\star t_x^\star \mathcal{M}^\vee}_{=:\mathcal{L}''}.$$

Now, exactly as in b) above we get a contradiction to Proposition 5.13: The first Chern class of $\mathcal{L}''$ is $a(H^\boxplus + s^\star H)$ and by assumption we have

$$\frac{f}{r} \leq 3a + b < 3a. \qquad \square$$

Proof of Theorem 5.18. First of all we can reduce to the case that $\mathcal{F}$ is a line bundle in the same way as in the proof of Theorem 4.6. Next, let $\mathcal{L} \subset \mathcal{F}^{[3]}$ be a destabilising line subbundle. As usual we see that this yields a nontrivial element in

$$\mathrm{Hom}(\mathcal{L}, \mathcal{F}^{[3]}) \cong \mathrm{Hom}(\mathcal{L}, \psi_{3\star} \sigma_3^\star q^\star \mathcal{F}) \cong \mathrm{Hom}(\psi_3^\star \mathcal{L}, \sigma_3^\star q^\star \mathcal{F}).$$

By the same reasoning as in Section 4.3 we may assume that $\mathcal{L} \simeq \mathcal{M}_{A^{[3]}} \otimes m_3^\star \mathcal{O}(bH))$, $c_1(\mathcal{M}) = aH$, $a, b \in \mathbb{Z}$ (we do not have any term of the form $\mathcal{O}(c\delta_3)$). Thus $\psi_3^\star \mathcal{L}$ descends to a line bundle $\mathcal{L}'$ on $A \times A^{[2]}$. Furthermore, the pullback $(\mathrm{id}_A \times \psi)^\star \mathcal{L}'$ on $A \times \widetilde{A \times A}$ descends to a line bundle $\mathcal{L}''$ on A^3. Thus we have

$$\begin{aligned} \mathrm{Hom}(\psi_3^\star \mathcal{L}, \sigma_3^\star q^\star \mathcal{F}) &= \mathrm{Hom}(\mathcal{L}', q^\star \mathcal{F}) \\ &\subseteq \mathrm{Hom}((\mathrm{id}_A \times \psi)^\star \mathcal{L}', (\mathrm{id}_A \times \sigma)^\star \pi_1^\star \mathcal{F}) \\ &= \mathrm{Hom}(\mathcal{L}'', \pi_1^\star \mathcal{F}). \end{aligned}$$

Looking more closely we see that $\mathcal{L}'' \simeq \mathcal{M}^{\boxtimes 3} \otimes s_3^\star \mathcal{O}(bH)$. Altogether we end up with a homomorphism on A^3:

$$\mathcal{M}^{\boxtimes 3} \otimes s_3^\star \mathcal{O}(bH) \to \pi_1^\star \mathcal{F}.$$

Now by Lemma 5.11 the destabilising condition on $\mathcal{L}$ reads

$$3a + b \geq f,$$

which exactly corresponds to the *tautological destabilising condition* of Proposition 5.19.

$\square$

5.5 Restriction to the Associated Kummer Surface

In Section 5.3 we proved the stability of tautological sheaves on the Hilbert scheme of two points on an abelian surface A. This Hilbert scheme contains the Kummer surface $\mathrm{Km}A$ associated with A. Recall that $\mathrm{Km}A$ is a $K3$ surface. In this section we shall prove the stability of the restriction of certain tautological sheaves to the Kummer surface.

Let (A, H) be a polarised abelian surface. In this section we do not restrict to principal polarisations. Also the Picard rank of A may be arbitrary. Recall the notation from Section 1.2: Denote by $b\colon \tilde{A} \to A$ the simultaneous blowup of all two-torsion points and by E_l, $l = 1, \ldots, 16$ the exceptional divisors. Furthermore, we have the quotient $\tau\colon \tilde{A} \to \mathrm{Km}A$ and we set $N_l = \tau(E_l)$. We want to compare the situation with the blowup diagram of Section 1.3 and consider this diagram:

$$\begin{tikzcd} A \arrow[r, hook, "u"] & A \times A \arrow[dr, "s"] & \\ \tilde{A} \arrow[u, "b"] \arrow[d, "\tau"] \arrow[r, hook, "\tilde{j}"] & \widetilde{A \times A} \arrow[u, "\sigma"] \arrow[d, "\psi"] \arrow[r] & A \\ \mathrm{Km}A \arrow[r, hook, "j"] & A^{[2]} \arrow[ur, "\rho"] & \end{tikzcd},$$

where we define $u(x) := (x, -x)$ and $\widetilde{j}$ as the pullback of u and σ. With this convention we have $\pi_1 \circ u = \mathrm{id}_A$. Finally, note that since ψ is flat, we have $j^\star \circ \psi_\star \simeq \tau_\star \circ \widetilde{j}^\star$.

Recall that we have a monomorphism $\alpha = \tau_! \circ b^\star\colon\ \mathrm{NS}(A) \to \mathrm{NS}(\mathrm{Km}A)$. Hence fixing a polarisation $H \in \mathrm{NS}\,A$, we can define a class

$$H_N^{\mathrm{Km}} := N\alpha(H) - \frac{1}{2}\sum_l N_l$$

on KmA, which is ample for sufficiently large N. Recall that on $A^{[2]}$ we defined two different polarisations $H_N := NH_{A^{[2]}} - \delta$ and $H_N^m := N(H_{A^{[2]}} + m^\star H) - \delta$.

Lemma 5.20. *We have*

$$H_N^{\mathrm{Km}} = j^\star H_N = j^\star H_N^m.$$

Proof. It is clear that $j^\star m^\star H = 0$ since KmA is a fibre of m. Also observe that the class of the intersection of the exceptional divisor of the Hilbert$-$Chow morphism ρ with KmA is exactly $\sum_l N_l$. $\square$

Definition 5.21. Let $\mathcal{F}$ be a sheaf on A. We set

$$\mathcal{F}^{\mathrm{Km}} := \tau_\star b^\star \mathcal{F}.$$

Lemma 5.22. *The sheaf $\mathcal{F}^{\mathrm{Km}}$ is the restriction of the tautological sheaf $\mathcal{F}^{[2]}$:*

$$j^\star \mathcal{F}^{[2]} \simeq \mathcal{F}^{\mathrm{Km}}.$$

Proof. We have

$$j^\star \mathcal{F}^{[2]} \;=\; j^\star \psi_\star \sigma^\star \pi_1^\star \mathcal{F} \;\simeq\; \tau_\star \widetilde{j}^\star \sigma^\star \pi_1^\star \mathcal{F} \;\simeq\; \tau_\star b^\star u^\star \pi_1^\star \mathcal{F} \;\simeq\; \tau_\star b^\star \mathcal{F} \;=\; \mathcal{F}^{\mathrm{Km}}. \qquad \square$$

Now we want to prove the stability of $\mathcal{F}^{\mathrm{Km}}$ in the case that $\mathcal{F}$ is of rank one or two. The method is completely analogous to the one used in the preceding sections. Thus we will leave a few details to the reader. As in the previous cases we begin with the analysis of line subbundles in the pullback $b^\star \mathcal{F}$:

Proposition 5.23. *Let $\mathcal{F}$ be a μ_H-stable sheaf on A of rank r and first Chern class $f \in \mathrm{NS}\,A$. Then $b^\star \mathcal{F}$ does not contain any line bundle $\mathcal{L}' = b^\star \mathcal{G} \otimes \mathcal{O}(\sum_l a_l E_l)$ with $\mathcal{G} \in \mathrm{Pic}(A)$, $c_1(\mathcal{G}) = g'$ satisfying*

$$H.g' \geq \frac{1}{r} H.f$$

but in the case $r = 1$, $\mathcal{G} \simeq \mathcal{F}$.

Proof. As usual we may assume that $\mathcal{F}$ is locally free. We want to show that

$$\mathrm{Hom}_{\tilde{A}}(b^\star \mathcal{G} \otimes \mathcal{O}(\sum_l a_l E_l), b^\star \mathcal{F})$$

vanishes. Using adjunction ($b^\star \dashv b_\star$) and a similar induction argument as in the proof of Lemma 4.1, we see that it is enough to prove that

$$\mathrm{Hom}_A(\mathcal{G}, \mathcal{F}) = 0.$$

This easily follows from the stability of $\mathcal{F}$ if $\mathcal{F} \not\simeq \mathcal{G}$. $\square$

Next, we will show that Proposition 5.23 implies that there are no destabilising line subbundles in $\mathcal{F}^{\mathrm{Km}}$. We only need to calculate slopes.

Lemma 5.24. *Let $\mathcal{F}$ be a sheaf on A of rank r and first Chern class f. We have*

$$c_1(\tau_\star b^\star \mathcal{F}) \;=\; \alpha(f) - \frac{r}{2}\sum_l N_l.$$

Proof. We have $c_1(\omega_{\tilde{A}}) = \sum_l E_l$. Thus the Grothendieck–Riemann–Roch theorem reads

$$\begin{aligned} \mathrm{ch}(\tau_\star b^\star \mathcal{F}) &= \tau_\star(\mathrm{ch}(b^\star\mathcal{F})\,\mathrm{td}_\tau) &&= \tau_\star((r, b^\star f, \dots)(1, -\tfrac{1}{2}\textstyle\sum_l E_l, \dots)) \\ & &&= \tau_\star(r, b^\star f - \tfrac{r}{2}\textstyle\sum_l E_l, \dots). \qquad \square \end{aligned}$$

Let $\mathcal{L}$ be a line bundle on $\mathrm{Km}A$. By equation (1) in Section 1.2 there is a line bundle $\mathcal{G}$ on A and integers a_l such that

$$\tau^\star\mathcal{L} \simeq b^\star\mathcal{G} \otimes \mathcal{O}(\sum_l a_l E_l).$$

Set $g := c_1(\mathcal{G})$. Note that since $\mathcal{L}$ comes from $\mathrm{Km}A$, the line bundle $\mathcal{G}$ has to be symmetric, i.e. $\iota^\star\mathcal{G} \simeq \mathcal{G}$.

Corollary 5.25. *Let $\mathcal{L}$ be a line bundle on* $\mathrm{Km}A$ *as above. We have*

$$\begin{aligned} \mu_{H_N^{\mathrm{Km}}}(\mathcal{F}^{\mathrm{Km}}) &= \frac{1}{r} NH.f - 4 \text{ and} \\ \mu_{H_N^{\mathrm{Km}}}(\mathcal{L}) &= NH.g + \frac{1}{2}\sum_l a_l. \end{aligned}$$

Proof. We pullback all classes to $\tilde{A}$: Note that $\tau^\star(\frac{1}{2}\sum_l N_l) = \sum_l E_l$ and $\tau^\star\alpha(f) = 2b^\star f$ for all $f \in \mathrm{NS}\,A$. Thus we have $\alpha(f).\alpha(H) = 2f.H$ as stated in Example 1.10. Furthermore, we have $\big(\sum_l E_l\big)^2 = 16\cdot(-1) = -16$ and $\big(\sum_l E_l\big)\big(\sum_l a_l E_l\big) = -\sum_l a_l$. Finally, we have to divide everything by two because we pulled back along a degree two covering. $\square$

Corollary 5.26. *Let $\mathcal{F}$ be a non-symmetric (i.e. $\iota^\star\mathcal{F} \not\simeq \mathcal{F}$) μ_H-stable sheaf on A. Then $\mathcal{F}^{\mathrm{Km}}$ does not contain $\mu_{H_N^{\mathrm{Km}}}$-destabilising line subbundles for all $N \gg 0$.*

Proof. Let $\mathcal{L}$ be a destabilising line subbundle of $\mathcal{F}^{\mathrm{Km}}$. Again, we can write $\tau^\star\mathcal{L} \simeq b^\star\mathcal{G} \otimes \mathcal{O}(\sum_l a_l E_l)$ for a symmetric line bundle $\mathcal{G} \in \mathrm{Pic}\,A$. The destabilising condition yields

$$H.g \geq \frac{1}{r}H.f.$$

As usal we use adjunction $\tau^\star \dashv \tau_\star$ to obtain a homomorphism $\tau^\star\mathcal{L} \to b^\star\mathcal{F}$. This gives a contradiction to Proposition 5.23 but in the case $r = 1$, $\mathcal{G} \simeq \mathcal{F}$. But this cannot be since $\mathcal{F}$ was chosen not to be symmetric. $\square$

We immediately deduce:

Theorem 5.27. *Let $\mathcal{F}$ be a non-symmetric rank one torsion-free sheaf on A. Then for all N sufficiently large, $\mathcal{F}^{\mathrm{Km}} = \tau_\star b^\star \mathcal{F}$ is a rank two $\mu_{H_N^{\mathrm{Km}}}$-stable sheaf.*

Example 5.28. We apply the theorem to the case $c_1(\mathcal{F}) = 0$. Denote by $\hat{A}$ the dual abelian variety and by $\hat{A}[2]$ its two-torsion points. The assignment $\operatorname{Pic}^0 A \ni \mathcal{F} \mapsto \mathcal{F}^{\mathrm{Km}}$ gives a map

$$\hat{A} \setminus \hat{A}[2] \to \mathcal{M},$$

where $\mathcal{M} := \mathcal{M}_{H_N^{\mathrm{Km}}}(v)$ is the moduli space of H_N^{Km}-stable sheaves with

$$v = (2, -\frac{1}{2}\sum_l N_l, -2).$$

Note that $v^2 = 0$ and the first Chern class $-\frac{1}{2}\sum_l N_l$ is primitive. Hence $\mathcal{M}$ is smooth of dimension two. Since $\mathcal{F}^{\mathrm{Km}} \simeq (\iota^\star\mathcal{F})^{[\mathrm{Km}]}$, this map is two-to-one. Furthermore, let us consider the case that $\mathcal{F}$ is symmetric, i.e. $\mathcal{F} \in \hat{A}[2]$. We concentrate on the case $\mathcal{F} = \mathcal{O}_A$. We have extensions

$$0 \to \mathcal{O}(-\frac{1}{2}\sum_l N_l) \to \mathcal{E} \to \mathcal{O} \to 0.$$

The sheaf $\mathcal{O}_X^{\mathrm{Km}}$ is isomorphic to the trivial extension $\mathcal{O}(-\frac{1}{2}\sum_l N_l) \oplus \mathcal{O}$ (cf. [BHPV04, Lem. 17.2]), which is not stable. On the other hand every nontrivial extension is μ_H-stable, which can be proven as in Lemma 7.6. The vector space of extensions $\mathcal{E}$ is two-dimensional and thus we have a $\mathbb{P}^1 \subset \mathcal{M}$ parametrising the $\mathcal{E}$. Altogether we see that $\mathcal{M}$ is isomorphic to the Kummer surface $\mathrm{Km}\hat{A}$ of the dual abelian surface $\hat{A}$.

If $\mathcal{F}$ has nontrivial first Chern class $f \in \operatorname{NS} A$, we may choose a symmetric line bundle $\mathcal{L}$ satisfying $c_1(\mathcal{L}) = -f$. Then $\mathcal{F} \otimes \mathcal{L}$ is in $\operatorname{Pic}^0(A)$ and

$$(\mathcal{F} \otimes \mathcal{L})^{[\mathrm{Km}]} \simeq \mathcal{F}^{\mathrm{Km}} \otimes \mathcal{O}(\alpha(-f)).$$

Thus the moduli space containing $\mathcal{F}^{\mathrm{Km}}$ is isomorphic to $\mathrm{Km}\hat{A}$, too.

But by [GH98, Thm. 1.5] the Kummer surfaces $\mathrm{Km}A$ and $\mathcal{M} \cong \mathrm{Km}\hat{A}$ are isomorphic.

We finish the section by proving the analogue of Theorem 4.8.

Theorem 5.29. *Let $\mathcal{F}$ be a μ_H-stable rank two sheaf on A such that* $\det \mathcal{F}$ *is not symmetric. Then $\mathcal{F}^{\mathrm{Km}}$ is a μ_{H_N}-stable rank four sheaf on* $\mathrm{Km}A$.

Proof. We exactly imitate the proof of Theorem 4.8. Assume that $\mathcal{F}$ is locally free and let $f := c_1(\mathcal{F})$. Let $\mathcal{E}$ be a reflexive semistable rank two subsheaf of $\mathcal{F}^{\mathrm{Km}}$ and write $c_1(\mathcal{E}) = \alpha(e) + \sum_l a_l N_l$. The destabilising condition thus implies

$$2H.e \geq H.f.$$

We have a homomorphism $\beta\colon \tau^\star\mathcal{E} \to b^\star\mathcal{F}$. Again, the only difficult case is when $\ker\beta = 0$: If the first Chern class of the $\mathcal{Q} := \operatorname{coker}\beta$ is trivial, we see that the homological dimension of $\mathcal{Q}$ is 2. Since $b^\star\mathcal{F}$ is locally free, this would contradict the fact that $\tau^\star\mathcal{E}$ is reflexive. Thus $\mathcal{Q} = 0$ and β has to be an isomorphism. But since $\tau^\star\mathcal{E}$ is symmetric and $\mathcal{F}$ is not, we are done.

If there is an effective divisor with first Chern class $c_1(\mathcal{Q})$, the line bundle

$$b^\star \det\mathcal{F} \otimes \tau^\star \det\mathcal{E}^\vee(-\sum_l a_l N_l)$$

must have a section. Hence either $a_l < 0\ \forall l$ and $\det\mathcal{F} \simeq \mathcal{O}_A$ (which we excluded) or $a_l \leq 0\ \forall l$ and

$$H.f > 2H.e$$

which contradicts the stability condition. □

5.6 Restriction to $K_3(A)$

Let (A, H) be a p.p.a.s. satisfying $(\star)$ and let $A^{[3]}$ be the Hilbert scheme of three points on A. In this section we prove some results concerning the stability of the restriction of tautological sheaves to the four dimensional generalised Kummer variety $j\colon K_3(A) \hookrightarrow A^{[3]}$.

We have an isomorphism

$$\mathrm{NS}(A^{[3]}) \cong \mathbb{Z}H_{A^{[3]}} \oplus \mathbb{Z}m_3^\star H \oplus \mathbb{Z}\delta_3.$$

Restricting to $K_3(A)$, we obtain

$$NS(K_3(A) \cong \mathbb{Z}j^\star H_{A^{[3]}} \oplus \mathbb{Z}\delta_3.$$

Note that again $j^\star m_3^\star H = 0$. Considering the polarisations $H_N = NH_{A^{[3]}} - \delta_3$ and $H_N^m = N(H_{A^{[3]}} + m_3^\star H) - \delta_3$ on $A^{[3]}$, we define a polarisation

$$H_N^K := j^\star H_N = j^\star H_N^m = Nj^\star H_{A^{[3]}} - j^\star\delta_3.$$

Lemma 5.30. *We have*

$$(H_N^K)^3 \cdot j^\star\delta_3 = 0 + O(N^4).$$

Proof. By definition of H_N^K we have $(H_N^K)^3 \cdot j^\star\delta_3 = j^\star((H_N)^3\delta_3)$. Now the lemma follows from equation (10) in Lemma 3.13. □

Proposition 5.31. *Let $\mathcal{F}$ be a μ_H-stable sheaf on A of rank r and first Chern class fH. If $\mathcal{F}^{\vee\vee} \not\simeq \mathcal{O}_A$, then for N sufficiently large $\mathcal{F}^{K_3} := j^\star \mathcal{F}^{[3]}$ does not contain any $\mu_{H_N^K}$-destabilising subsheaves of rank one.*

Proof. We may assume that $\mathcal{F}$ is locally free. Since all line bundles on $K_3(A)$ come from $A^{[3]}$, we may assume that a destabilising line bundle is of the form $j^\star\mathcal{M}'$ with $\mathcal{M}' \in \operatorname{Pic}(A^{[3]})$. Furthermore, since $j^\star\mu_3^\star H = 0$, we may assume that we have no contribution of this summand in $\mathcal{M}'$ and by a similar reasoning as in Lemma 4.1 we can reduce to the case where we have no contribution of the δ_3-summand neither. Thus there is, in fact, a line bundle $\mathcal{M}$ on A such that $\mathcal{M}' \simeq \mathcal{M}_{A^{[3]}}$. Let $l \in \mathbb{Z}$ be such that $lH = c_1(\mathcal{M})$.

We consider the following two diagrams:

$$\begin{tikzcd}
A^{[2,3]} \arrow[r, "\sigma_3"] \arrow[d, "\psi_3"'] & A \times A^{[2]} \arrow[d] & A \times \widetilde{A \times A} \arrow[l, "\psi"'] \arrow[d, "\sigma"] \\
A^{[3]} \arrow[r, "\rho_3"] \arrow[dr, "m_3"'] & S^3A \arrow[d, "\tilde{s}_3"] & A^3 \arrow[l] \arrow[dl, "s_3"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\widetilde{K_3(A)}, \widetilde{j}) \arrow[r, "\sigma_3'"] \arrow[d, "\psi_3'"'] & (B,k) \arrow[d] & (\widetilde{B}, \widetilde{k}) \arrow[l, "\psi'"'] \arrow[d, "\sigma'"] \\
(K_3(A), j) \arrow[r] \arrow[dr] & \tilde{s}_3^{-1}(0) \arrow[d] & (s_3^{-1}(0), g) \arrow[l] \arrow[dl] \\
& \{0\} &
\end{tikzcd}.$$

The right diagram is a subdiagram of the left one. Where needed the symbols for the inclusion morphisms are added. For example, (B,k) denotes $B := \{(x,\xi) \mid x + m(\xi) = 0\}$ together with the inclusion $k \colon B \hookrightarrow A \times A^{[2]}$ and $\widetilde{K_3(A)}$ is the strict transform of B along σ_3. In the left diagram we used the abreviations ψ and σ to denote $\mathrm{id}_A \times \psi$ and $\mathrm{id}_A \times \sigma$, respectively.

Note that ψ_3 is flat outside codimension four and thus its restriction ψ_3', too, because $\widetilde{K_3(A)}$ is the preimage of $K_3(A)$ under ψ_3. Thus for all sheaves $\mathcal{G}$ on $K_3(A)$ and $\mathcal{H}$ on $A^{[2,3]}$ we have an isomorphism

$$\operatorname{Hom}(\mathcal{G}, j^\star\psi_{3\star}\mathcal{H}) \;\cong\; \operatorname{Hom}(\mathcal{G}, \psi'_{3\star}\widetilde{j}^\star\mathcal{H}).$$

Furthermore, we have

$$\psi_3'^\star j^\star \simeq \widetilde{j}^\star\psi_3^\star, \quad \widetilde{j}^\star\sigma_3^\star \simeq \sigma_3'^\star k^\star, \quad \psi'^\star k^\star \simeq \widetilde{k}^\star\psi^\star \quad \text{and} \quad \widetilde{k}^\star\sigma^\star \simeq \sigma'^\star g^\star.$$

Finally, we have

$$\begin{aligned} \psi_3^\star\mathcal{M}_{A^{[3]}} &\simeq \sigma_3^\star(\mathcal{M} \boxtimes \mathcal{M}_{A^{[2]}}) \quad \text{and} \\ \psi^\star(\mathcal{M} \boxtimes \mathcal{M}_{A^{[2]}}) &\simeq \sigma^\star\mathcal{M}^{\boxtimes 3}. \end{aligned}$$

We denote the projections $A^3 \to A$ by π_i. We have

$$
\begin{array}{rlcl}
& \mathrm{Hom}(j^\star\mathcal{M}, j^\star\mathcal{F}^{[3]}) & \cong & \mathrm{Hom}(j^\star\mathcal{M}, j^\star\psi_{3\star}\sigma_3^\star\pi_1^\star\mathcal{F}) \\
\cong & \mathrm{Hom}(j^\star\mathcal{M}, \psi'_{3\star}\widetilde{j}^\star\sigma_3^\star\pi_1^\star\mathcal{F}) & \cong & \mathrm{Hom}(\psi_3'^\star j^\star\mathcal{M}, \widetilde{j}^\star\sigma_3^\star\pi_1^\star\mathcal{F}) \\
\cong & \mathrm{Hom}(\widetilde{j}^\star\psi_3^\star\mathcal{M}, \widetilde{j}^\star\sigma_3^\star\pi_1^\star\mathcal{F}) & \cong & \mathrm{Hom}(\widetilde{j}^\star\sigma_3^\star(\mathcal{M}\boxtimes\mathcal{M}_{A^{[2]}}), \widetilde{j}^\star\sigma_3^\star\pi_1^\star\mathcal{F}) \\
\cong & \mathrm{Hom}(\sigma_3'^\star k^\star(\mathcal{M}\boxtimes\mathcal{M}_{A^{[2]}}), \sigma_3'^\star k^\star\pi_1^\star\mathcal{F}) & \cong & \mathrm{Hom}(k^\star(\mathcal{M}\boxtimes\mathcal{M}_{A^{[3]}}), k^\star\pi_1^\star\mathcal{F}) \\
\subseteq & \mathrm{Hom}(\psi'^\star k^\star(\mathcal{M}\boxtimes\mathcal{M}_{A^{[3]}}), \psi'^\star k^\star\pi_1^\star\mathcal{F}) & \cong & \mathrm{Hom}(\widetilde{k}^\star\psi^\star(\mathcal{M}\boxtimes\mathcal{M}_{A^{[3]}}), \widetilde{k}^\star\psi^\star\pi_1^\star\mathcal{F}) \\
\cong & \mathrm{Hom}(\widetilde{k}^\star\sigma^\star\mathcal{M}^{\boxtimes 3}, \widetilde{k}^\star\sigma^\star\pi_1^\star\mathcal{F}) & \cong & \mathrm{Hom}(\sigma'^\star g^\star\mathcal{M}^{\boxtimes 3}, \sigma'^\star g^\star\pi_1^\star\mathcal{F}) \\
\cong & \mathrm{Hom}(g^\star\mathcal{M}^{\boxtimes 3}, g^\star\pi_1^\star\mathcal{F}) & \cong & \mathrm{H}^0(g^\star(\mathcal{M}^{\vee\boxtimes 3}\otimes\pi_1^\star\mathcal{F})).
\end{array}
$$

In order to proceed, we choose an isomorphism $s_3^{-1}(0) \cong A^2$ by sending (x,y,z) to (x,z) and denote the projections $A^2 \to A$ by $\hat{\pi}_i$. In this picture we have the identifications: $\pi_1 \circ g = \hat{\pi}_1$, $\pi_2 \circ g \mathrel{\widehat{=}} \iota \circ s$ and $\pi_3 \circ g \mathrel{\widehat{=}} \hat{\pi}_2$. (Recall that $s\colon A^2 \to A$ denotes the summation map.) Thus pushing forward along p_1 (p_2 in the second line), we have

$$
\begin{aligned}
\mathrm{H}^0(g^\star(\mathcal{M}^{\vee\boxtimes 3}\otimes\pi_1^\star\mathcal{F})) &\cong \mathrm{H}^0(\mathcal{F}\otimes\mathcal{M}^\vee\otimes\hat{\pi}_{1\star}(\hat{\pi}_2^\star\mathcal{M}^\vee\otimes s^\star\iota^\star\mathcal{M}^\vee)) && (25)\\
&\cong \mathrm{H}^0(\mathcal{M}^\vee\otimes\hat{\pi}_{2\star}(\hat{\pi}_1^\star(\mathcal{F}\otimes\mathcal{M}^\vee)\otimes s^\star\iota^\star\mathcal{M}^\vee)). && (26)
\end{aligned}
$$

By Lemma 5.30 the destabilising condition of $j^\star\mathcal{M}$ in $j^\star\mathcal{F}^{[3]}$ implies

$$l \geq \frac{f}{3r}.$$

If $l \geq 0$, we see that the right hand side of (25) vanishes but in the case $\mathcal{M} \simeq \mathcal{O}_A \simeq \mathcal{F}$. If $l < 0$, the destabilising condition implies

$$2l > \frac{f}{r}.$$

In this case the right hand side of (26) has to vanish by Proposition 5.13. □

As usual, from Proposition 5.31 we can deduce the stability of rank three restricted tautological sheaves associated with rank one sheaves:

Theorem 5.32. *Let $\mathcal{F}$ be a torsion-free rank one sheaf on A. Assume $\det\mathcal{F} \not\simeq \mathcal{O}_A$. Then for all sufficiently large N the sheaf $j^\star\mathcal{F}^{[3]}$ is μ_N^K-stable.*

6 Deformations and Moduli Spaces of Tautological Sheaves

In the preceding two chapters we have proven many results concerning the stability of tautological sheaves on Hilbert schemes of surfaces. A crucial assumption was always that the sheaf $\mathcal{F}$ on the surface was stable (which is automatically satisfied for every torsion-free rank one sheaf). In this chapter we want to study deformations of tautological sheaves and the relation between the moduli spaces of $\mathcal{F}$ and $\mathcal{F}^{[n]}$. We will only consider the case when X is a (projective) $K3$ surface and mainly focus on the case $n = 2$.

The choice of polarisations was crucial for the proof of stability in Chapters 4 and 5. In order to minimise the complexity of the notation in this chapter, we will refrain from mentioning the polarisations throughout.

6.1 Deformations of Tautological Sheaves

In this section we will make the following general assumption:

$$\left(\begin{array}{c} X \text{ is a } K3 \text{ surface and } \mathcal{F} \text{ a stable sheaf on } X \text{ with Mukai vector } v \text{ such that} \\ \text{for every sheaf } \mathcal{G} \in \mathcal{M}^s(v) \text{ the associated tautological sheaf } \mathcal{G}^{[n]} \text{ is also stable.} \end{array}\right)$$

Note that in the cases where the stability of tautological sheaves has been explicitly proven the tautological sheaf associated with a sheaf $\mathcal{F}$ is stable if and only if it is true for every other $\mathcal{G}$ in the same moduli space. (We are only considering sheaves on $K3$ surfaces.)

Denote by $v^{[n]} \in \mathrm{H}^*(X^{[n]}, \mathbb{Q})$ the Mukai vector of $\mathcal{F}^{[n]}$. The assignment

$$\mathcal{F} \mapsto \mathcal{F}^{[n]}$$

yields a morphism

$$[-]^{[n]} \colon \mathcal{M}^s(v) \to \mathcal{M}^s(v^{[n]}).$$

We shall mainly discuss the case $n = 2$. Let us prove the following lemma which shows that $[-]^{[2]}$ is injective on closed points.

Lemma 6.1. *For every sheaf $\mathcal{F}$ on X we have*

$$\mathcal{F} \simeq \mathcal{T}or^1_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \sigma_\star\psi^\star\mathcal{F}^{[2]}).$$

Thus we can reconstruct the original sheaf $\mathcal{F}$ from the tautological sheaf $\mathcal{F}^{[2]}$.

Proof. Recall that we have an exact sequence on $X \times X$ (cf. Propostion 3.7):

$$0 \to \sigma_\star\psi^\star\mathcal{F}^{[2]} \to \mathcal{F}^{\boxplus 2} \to \Delta_\star\mathcal{F} \to 0. \tag{27}$$

We tensor this sequence with the structure sheaf of the diagonal $\Delta \subset X \times X$. Of course we have

$$\pi_1^\star \mathcal{F}|_\Delta \simeq \Delta^\star \pi_1^\star \mathcal{F} \simeq \mathcal{F}$$

and the higher Tors $\mathcal{T}or^i_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \pi_1^\star \mathcal{F})$ vanish. Therefore we have an isomorphism

$$\mathcal{T}or^1_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \sigma_\star \psi^\star \mathcal{F}^{[2]}) \simeq \mathcal{T}or^2_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \Delta_\star \mathcal{F}).$$

By Proposition 11.8 in [Huy06] we find

$$\mathcal{T}or^i_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \Delta_\star \mathcal{F}) = \begin{cases} \mathcal{F} & i = 0, 2 \text{ and} \\ \mathcal{F} \otimes \Omega_X & i = 1. \end{cases} \qquad \square$$

Remark 6.2. If we tensor (27) with $\mathcal{O}_\Delta$ as above, the first terms of the resulting long exact Tor-sequence yield a short exact sequence

$$0 \to \mathcal{F} \otimes \Omega_X \to \sigma_\star \psi^\star \mathcal{F}^{[2]}|_\Delta \to \mathcal{F} \to 0.$$

It is not clear if this exact sequence is split or if it is equivalent to the natural extension corresponding to the Atiyah class of $\mathcal{F}$.

Let us consider a stable sheaf $\mathcal{F}$ on a $K3$ surface X. The stability implies that either $h^0(X, \mathcal{F})$ or $h^2(X, \mathcal{F}) = h^0(X, \mathcal{F}^\vee)$ vanishes. Let us assume the former is the case. (The case $h^2(X, \mathcal{F}) = 0$ can be treated in exactly the same way.) Corollary 3.10 shows that we have a natural monomorphism

$$[-]^{[2]} \colon \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) \hookrightarrow \operatorname{Ext}^1(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}),$$

which maps an infinitesimal deformation of $\mathcal{F}$ to its induced deformation of $\mathcal{F}^{[2]}$.

Definition 6.3. We call an infinitesimal deformation of $\mathcal{F}^{[2]}$, the class of which lies in the image of $[-]^{[2]}$ above, a *surface deformation.* Deformations lying in the other summand of equation (8) in Corollary 3.10 are referred to as *additional deformations.*

We conclude:

Proposition 6.4. *We have an embedding of moduli spaces*

$$\mathcal{M}^s(v) \hookrightarrow \mathcal{M}^s(v^{[2]}).$$

The additional deformations are isomorphic to $\mathrm{H}^0(X, \mathcal{F}) \otimes \mathrm{H}^1(X, \mathcal{F})^\vee$.

Corollary 6.5. *Let $\mathcal{F}$ be such that $h^1(X, \mathcal{F}) = 0$. Then we have a local isomorphism of the corresponding moduli spaces.*

Corollary 6.6. *Let $\mathcal{F}$ be such that $\mathcal{M}^s(v)$ is compact and $h^1(X, \mathcal{G}) = 0$ for all $\mathcal{G} \in \mathcal{M}^s(v)$. Then we have an isomorphism of $\mathcal{M}^s(v)$ with a connected component of $\mathcal{M}^s(v^{[2]})$.*

6.2 The Additional Deformations and Singular Moduli Spaces

In the last section we have seen that the surface deformations of tautological sheaves are unobstructed. This is not true for all deformations. Indeed, in this section we will give an explicit construction of an example of a sheaf $\mathcal{F}$ on an elliptically fibred $K3$ surface such that $\mathcal{F}^{[2]}$ is stable and the corresponding point in the moduli space is singular.

To prove this statement let us recall the most basic properties of the Kuranishi map: The general idea of the deformation theory of a stable sheaf $\mathcal{F}$ is that infinitesimal deformations are parametrised by $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$ and the obstructions lie in $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})$. This is formalised by the so-called Kuranishi map. More precisely it can be shown that there is a map $\kappa\colon \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \to \mathrm{Ext}^2(\mathcal{F},\mathcal{F})$ such that the completion of the local ring of the point of the moduli space corresponding to $\mathcal{F}$ is isomorphic to the local ring of $\kappa^{-1}(0)$ in 0. In general there is no direct geometric description of the Kuranishi map but it is known that the constant and linear terms of the power series expansion of κ vanish and that its quadratic part is given by $\kappa_2\colon \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \to \mathrm{Ext}^2(\mathcal{F},\mathcal{F}), e \mapsto \frac{1}{2}(e \circ e)$.

For a $K3$ surface this quadratic term always vanishes since it is exactly the Serre duality pairing which is known to be alternating. But if we consider a tautological sheaf $\mathcal{F}^{[2]}$ the quadratic part of the Kuranishi map may be non-trivial. This would correspond to the existence of a quadratic part in the equation of the tangent cone of the point in the moduli space corresponding to $\mathcal{F}^{[2]}$. Consequently, the tangent cone would be strictly smaller than the tangent space and we would end up with a singularity.

Example 6.7. Let X be an elliptically fibred $K3$ surface with fibre class E and section C. Consider the line bundle $\mathcal{G} := \mathcal{O}(kF)$, $k \geq 2$. We have $h^0(\mathcal{G}) = k+1$ and $h^1(\mathcal{G}) = k-1$. Certainly $\mathcal{G}$ is stable and the moduli space is a reduced point. The rank two tautological sheaf $\mathcal{G}^{[2]}$ is also stable and the tangent space of its moduli space at the point corresponding to $\mathcal{G}^{[2]}$ is isomorphic to $\mathrm{H}^0(X,\mathcal{G}) \otimes \mathrm{H}^1(X,\mathcal{G})^\vee$, which has dimension k^2-1. The quadratic term of the Kuranishi map vanishes identically but it is not clear if we can deform $\mathcal{G}^{[2]}$ along any of these infinitesimal directions.

Example 6.8. We continue with the same elliptic $K3$ as above. From [DM89] we learn that the linear system of the line bundle $\mathcal{L}$ with first Chern class $C + kE$ has C as a base component for $k \geq 2$. We have $h^0(\mathcal{G}) = k+1$ and $h^1(\mathcal{G}) = 0$. Now let p be a point on the curve C and denote by $\mathcal{I}_p$ the corresponding ideal sheaf. We set $\mathcal{F} := \mathcal{L} \otimes \mathcal{I}_p$. Certainly $\mathcal{F}$ is a torsion-free rank one sheaf with nonvanishing first Chern class. Hence $\mathcal{F}^{[2]}$ is stable by Theorem 4.6.

Theorem 6.9. *The point in the moduli space corresponding to $\mathcal{F}^{[2]}$ is singular.*

By the above considerations we have to prove the following lemma:

Lemma 6.10. *For the example $\mathcal{F}^{[2]} = (\mathcal{L} \otimes \mathcal{I}_p)^{[2]}$ the quadratic part of the Kuranishi map does not vanish.*

Proof. We have to analyse the Yoneda square

$$\begin{array}{ccc} \operatorname{Ext}^1(\mathcal{F}^{[2]},\mathcal{F}^{[2]}) & \to & \operatorname{Ext}^2(\mathcal{F}^{[2]},\mathcal{F}^{[2]}). \\ x & \mapsto & x \circ x \end{array}$$

Therefore let us use Krug's formula (8) in Corollary 3.10 to write down the extension groups explicitly. Note that $h^2(\mathcal{F}) = 0$.

$$\begin{array}{rcl} \operatorname{Ext}^1(\mathcal{F}^{[2]},\mathcal{F}^{[2]}) & \cong & \operatorname{Ext}^1(\mathcal{F},\mathcal{F}) \bigoplus \mathrm{H}^1(\mathcal{F})^\vee \otimes \mathrm{H}^0(\mathcal{F}), \\ \operatorname{Ext}^2(\mathcal{F}^{[2]},\mathcal{F}^{[2]}) & \cong & \operatorname{Ext}^2(\mathcal{F},\mathcal{F}) \bigoplus \mathrm{H}^0(\mathcal{F})^\vee \otimes \mathrm{H}^0(\mathcal{F}) \bigoplus \mathrm{H}^1(\mathcal{F})^\vee \otimes \mathrm{H}^1(\mathcal{F}). \end{array}$$

According to this decomposition we can decompose the Yoneda square as well following the detailed formulas in [Kru11, Sect. 7]:

$$\begin{array}{ccccccc} \operatorname{Ext}^1(\mathcal{F},\mathcal{F}) & \bigoplus & \mathrm{H}^1(\mathcal{F})^\vee \otimes \mathrm{H}^0(\mathcal{F}) & \to & & & \\ e & + & a \otimes b & \mapsto & & & \\ & & & & & & \\ \operatorname{Ext}^2(\mathcal{F},\mathcal{F}) & \bigoplus & \mathrm{H}^0(\mathcal{F})^\vee \otimes \mathrm{H}^0(\mathcal{F}) & \bigoplus & \mathrm{H}^1(\mathcal{F})^\vee \otimes \mathrm{H}^1(\mathcal{F}). & & \\ \underbrace{e \circ e}_{=0} & + & (a \circ e) \otimes b & + & a \otimes (e \circ b) & & \end{array}$$

Hence we need to show that the map

$$\operatorname{Ext}^1(\mathcal{F},\mathcal{F}) \times \mathrm{H}^0(\mathcal{F}) \to \mathrm{H}^1(\mathcal{F})$$

is not the zero map. The geometric interpretation of this map is the following: Let $e \in \operatorname{Ext}^1(\mathcal{F},\mathcal{F})$ be an infinitesimal deformation of $\mathcal{F}$ and $\varphi \in \mathrm{H}^0(\mathcal{F})$ be a global section. Then $\varphi \circ e$ is zero if and only if we can deform the section φ along e.

It is time to return to the geometry of our example. Since p is on the base curve C, we have $\mathrm{H}^0(\mathcal{F}) \cong \mathrm{H}^0(\mathcal{L})$. The deformations of $\mathcal{F}$ are those of $\mathcal{I}_p$, which correspond to deforming the point p in X. Now if we deform p into a direction normal to C, the space of global sections will shrink since the point will fail to be a base point of $\mathcal{L}$ and thus we can find a section $\varphi \in \mathrm{H}^0(\mathcal{F})$ that does not deform with e. □

The Zariski tangent space is $(k+3)$-dimensional and we can explicitly derive the quadratic equation of the tangent cone. It is equivalent to the intersection of a plane (corresponding to the surface deformations) and a hyperplane (the additional deformations and the curve C) in a line (the curve C).

6.3 Deformations of the manifold $X^{[n]}$

A question which has not been touched so far, is the following: The manifold $X^{[n]}$ has an unobstructed deformation theory. Does the tautological sheaf $\mathcal{F}^{[n]}$ deform with $X^{[n]}$?

The technique to answer this question is presented in [HT10]. We can summarise as follows:

Theorem 6.11 (Huybrechts–Thomas)**.** *Let Y be a projective manifold and $\mathcal{E}$ a sheaf on Y. Let $\kappa \in \mathrm{H}^1(Y, \mathcal{T}_Y) \cong \mathrm{Ext}^1(\Omega_Y, \mathcal{O}_Y)$ be the Kodaira–Spencer class of an infinitesimal deformation of Y and denote by $\mathrm{At}(\mathcal{E}) \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_Y)$ the Atiyah class of $\mathcal{E}$. The sheaf $\mathcal{E}$ can be deformed along κ if and only if*

$$0 = \mathrm{ob}(\kappa, \mathcal{E}) := (\kappa \otimes \mathrm{id}_{\mathcal{E}}) \circ \mathrm{At}(\mathcal{E}) \in \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}).$$

For every sheaf $\mathcal{E}$ on Y there are natural trace maps

$$\begin{aligned} \mathrm{tr}\colon \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_Y) &\to \mathrm{H}^1(Y, \Omega_Y) \quad \text{and} \\ \mathrm{tr}\colon \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) &\to \mathrm{H}^2(Y, \mathcal{O}_Y), \end{aligned}$$

which — up to a sign — commute with the Yoneda product. Furthermore, it is well known that

$$\begin{aligned} \mathrm{tr}(\mathrm{At}(\mathcal{E})) &= c_1(\mathcal{E}) \quad \text{and} \\ \mathrm{tr}(\mathrm{ob}(\kappa, \mathcal{E})) &= \mathrm{ob}(\kappa, \det \mathcal{E}) \end{aligned}$$

Applying this theorem to our situation, we get the following picture: The tangent space of the Kuranishi space at the point corresponding to $X^{[n]}$ is isomorphic to

$$\mathrm{H}^1(X^{[n]}, \mathcal{T}_{X^{[n]}}) \cong \mathrm{H}^1(X^{[n]}, \Omega_{X^{[n]}}) \cong \mathrm{H}^1(X, \Omega_X) \oplus \mathbb{C}\delta_n.$$

We write a class in $\mathrm{H}^1(X^{[n]}, \Omega_{X^{[n]}})$ as (κ, a) with $\kappa \in \mathrm{H}^1(X, \Omega_X)$ the class of an infinitesimal deformation of the surface X and $a \in \mathbb{C}$. Unfortunately there is no decomposition of the Atiyah class $\mathrm{At}(\mathcal{F}^{[n]})$ at hand. But we can at least study its trace $\mathrm{tr}(\mathrm{At}(\mathcal{F}^{[n]})) = c_1(\mathcal{F}^{[n]}) = c_1(\mathcal{F})_{X^{[n]}} - r\delta_n$, where we set $r := \mathrm{rk}\,\mathcal{F}$. We have:

$$\mathrm{tr}(\mathrm{ob}((\kappa, a), \mathcal{F}^{[n]})) = \mathrm{ob}(\kappa, \det \mathcal{F}) - ra\delta_n^2 \in \mathrm{H}^2(X^{[n]}, \mathcal{O}_{X^{[n]}}).$$

But we have $\mathrm{ob}(\kappa, \det \mathcal{F}) = \kappa \cdot c_1(\mathcal{F})$ and $\delta_n^2 = 2(1-n)$, where we consider the Beauville–Bogomolov pairing. Thus we see:

- If $\mathcal{F}$ deforms along κ, then surely the tautological sheaf $\mathcal{F}^{[n]}$ deforms along $(\kappa, 0)$.
- If the determinant line bundle $\det \mathcal{F}$ does not deform along κ, then $\mathcal{F}^{[n]}$ does not deform along $(\kappa, 0)$.
- If $\kappa \cdot c_1(\mathcal{F}) \neq 2(1-n)ra$, the tautological sheaf $\mathcal{F}^{[n]}$ does not deform along (κ, a).

Thus there is an interesting hyperplane inside the space of infinitesimal deformations of $X^{[n]}$ consisting of all pairs (κ, a) such that $\kappa \cdot c_1(\mathcal{F}) = 2(1-n)ra$: It is an open question if the tautological sheaf deforms along these directions.

7 Non-symplectic Involutions on Manifolds of $K3^{[2]}$-type

In this Chapter we study non-symplectic involutions on manifolds which are deformation equivalent to the Hilbert scheme of two points on a $K3$ surface. In Section 7.1 we recall Beauville's classification of 19-dimensional families of non-symplectic involutions on manifolds of $K3^{[2]}$-type and give a refinement of this classifcation by means of lattice theory. The new non-natural example will be constructed in Section 7.2 and we analyse its fixed locus in Section 7.3.

7.1 19-dimensional families of non-symplectic involutions

In [Beau11] Beauville gave a rough classification of non-symplectic involutions on manifolds of $K3^{[2]}$-type and proved that the fixed locus $F = \cup F_i$ is the union of smooth Lagrangian surfaces.

Theorem 7.1 (Beauville)**.** *Let Y be of $K3^{[2]}$-type, ι a non-symplectic involution on Y and F the fixed locus. Let t be the trace of ι^* acting on $\mathrm{H}^{1,1}(Y)$.*

1. *We have $\sum_i K_{F_i}^2 = t^2 - 1$, $\quad \sum_i \chi(\mathcal{O}_{F_i}) = \frac{1}{8}(t^2+7)$ $\quad$ and $\quad \sum_i e(F_i) = \frac{1}{2}(t^2+23)$.*
2. *The local deformation space of (Y, ι) is smooth of dimension $\frac{1}{2}(21-t)$.*
3. *The integer t takes any odd value between -19 and 21.*

Proof. [Beau11, Thm. 2]. □

The maximal dimension of a family of non-symplectic involutions is thus 20. In fact, the double covers of EPW-sextics by O'Grady [O'G06] constitute a locally complete family of manifolds of $K3^{[2]}$-type together with involutions (given by the covering transformations).

In the 19-dimensional case the invariants of F are

$$\sum_i K_{F_i}^2 = 288, \quad \sum_i \chi(\mathcal{O}_{F_i}) = 37 \quad \text{and} \quad \sum_i e(F_i) = 156.$$

An example is discussed in [Beau11]: Let $X \to \mathbb{P}^2$ be a double cover branched along a smooth curve $C \subset \mathbb{P}^2$ of degree six and genus ten. Let φ be the covering involution acting on X. It is well known that φ is non-symplectic and it induces a non-symplectic involution $\iota = \varphi^{[2]}$ on $Y = X^{[2]}$. There is a 19-dimensional family of these double sextics $X \to \mathbb{P}^2$ and thus we obtain a 19-dimensional family of pairs $(X^{[2]}, \varphi^{[2]})$. The involution $\varphi^{[2]}$ is called *natural* following [Boi12]. Note that the fixed locus of $\varphi^{[2]}$ is the union of the symmetric square $C^{[2]}$ and the quotient $\mathbb{P}^2 = X/\varphi$.

In the recent preprint [OW13] we give a finer classification of 19-dimensional families of non-symplectic involutions on manifolds of $K3^{[2]}$-type using lattice theory. Let Y be a manifold of $K3^{[2]}$-type and ι a non-symplectic involution such that the trace t is equal to -17. The invariant lattice

$$\mathrm{H}^2(Y,\mathbb{Z})^\iota$$

has rank two and is hyperbolic. We need one definition to be able to state the precise result of the classification.

Definition 7.2. For a non-degenerate lattice $(L,(\ ,\ ))$ and $l \in L$, we define $\mathrm{div}_L(l) \in \mathbb{Z}_{\geq 0}$ to be the positive number(!) generating the ideal

$$(\mathrm{div}_L(l))\mathbb{Z} = (l, L) = \{(l, l') \in \mathbb{Z} \mid l' \in L\}$$

and call it the *divisor of l in L*.

Recall that the second cohomology of Y is isomorphic to

$$\Lambda := U^3 \oplus E_8(-1)^2 \oplus \langle -2 \rangle.$$

Theorem 7.3. *Let ι be an involution acting on Y such that the fixed sublattice Λ^ι is hyperbolic of rank two. Such ι are divided into four conjugacy classes in $O(\Lambda)$. They are distinguished by the properties of Λ^ι as follows.*

No.	*isom. class of* Λ^ι	*property*
1	U	
2	$U(2)$	
3	$\langle 2 \rangle \oplus \langle -2 \rangle$	$\mathrm{div}_\Lambda(g) = 2$
4	$\langle 2 \rangle \oplus \langle -2 \rangle$	$\mathrm{div}_\Lambda(g) = 1$

In No. 3 *and* 4, *g denotes the generator of $\langle -2 \rangle$ (which is unique up to a sign).*

Furthermore, in the cases 1, 2, *and* 4 *there is exactly one* 19*-dimensional family of biregular involutions of the given types.*

Remark 7.4. The proof of Theorem 7.3 is due to H. Ohashi and can be found in [OW13].

The natural involution $\varphi^{[2]}$ acting on the Hilbert scheme of two points has the lattice structure corresponding to No. 3. This will be shown in the next section together with the construction of a new example realising No. 1. At this point we do not know how to realise the remaining two types geometrically.

7.2 Construction of a New Non-natural Involution

In this section we construct a 19-dimensional family of moduli spaces of sheaves on $K3$ surfaces using the methods of Section 2.4 and prove that it is a new example by means of the lattice theory developed in the preceding section.

Let $\pi\colon X \to \mathbb{P}^2$ be a double cover branched along a smooth sextic curve C. This construction yields a 19 dimensional family of $K3$ surfaces X with involution φ given by exchanging the covering sheets. Assume that $\operatorname{Pic}(X) \cong \mathbb{Z}H$, where H is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ (thus $H^2 = 2$). The pullback of a general line $l \subset \mathbb{P}^2$ is a smooth genus two curve in the linear system $|\mathcal{O}(H)| \cong |\mathcal{O}_{\mathbb{P}^2}(1)|$, the dual projective plane. Furthermore, from $\operatorname{Pic}(X) \cong \mathbb{Z}H$ it follows that we only have three kinds of degenerations: If l is tangent (bitangent) to C, the pullback is an elliptic (rational) curve with one (two) ordinary double point(s) and if l is tangent to C in an inflection point, the pullback is an elliptic curve with an ordinary triple point. The sextic C cannot have triple tangents since the pullback of such a line would split into two smooth rational curves in X which span a rank two lattice inside $\operatorname{Pic} X$. Note that, in particular, all curves in the linear system $|\mathcal{O}(H)|$ are reduced and irreducible. The locus of X having Picard rank one is the complement of a countable union of closed subvarieties inside the moduli space of polarised $K3$ surfaces. And finally, it is well known that the involution φ is non-symplectic. (If φ preserved the symplectic form, the quotient would have to be symplectic as well.)

The involution φ induces the natural involution $\varphi^{[2]}$ on the corresponding Hilbert scheme of two points $X^{[2]}$. We can regard $X^{[2]}$ as the moduli space of ideal sheaves of two points on X. Recall that for any sheaf $\mathcal{F}$ of rank r and Chern classes c_1 and c_2 we define its Mukai vector by

$$v(\mathcal{F}) := \operatorname{ch}(\mathcal{F})\sqrt{\operatorname{td}_X} = (r, c_1, c_1^2/2 - c_2 + r).$$

This vector is an element of the Mukai lattice which as an abelian group is isomorphic to

$$\mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^2(X,\mathbb{Z}) \oplus \mathrm{H}^4(X,\mathbb{Z}).$$

The Mukai vector of the ideal sheaf of two points can be computed as $(1, 0, -1)$. By Lemma 2.38 we see that the invariant lattice $\mathrm{H}^2(X^{[2]},\mathbb{Z})^\iota$ is spanned by $(0, H, 0)$ and $(1, 0, 1)$. Thus it is isomorphic to $\langle 2\rangle \oplus \langle -2\rangle$. The second summand corresponds to the exceptional divisor in $X^{[2]}$.

Now we come to the construction of the new example. Consider a length three subscheme $Z \subset X$ with ideal sheaf $\mathcal{I}_Z$.

Lemma 7.5. *We have*

$$h^1(\mathcal{I}_Z(H)) = \begin{cases} 1 & \text{if } Z \text{ lies on a curve } D_Z \in |\mathcal{O}(H)|, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have a short exact sequence

$$0 \to \mathcal{I}_Z(H) \to \mathcal{O}_X(H) \to \mathcal{O}_Z \to 0.$$

The corresponding long exact sequence of cohomology starts with

$$0 \to \mathrm{H}^0(\mathcal{I}_Z(H)) \to \mathrm{H}^0(\mathcal{O}_X(H)) \to \mathrm{H}^0(\mathcal{O}_Z) \to \mathrm{H}^1(\mathcal{I}_Z(H)) \to 0.$$

Both terms in the middle are three dimensional and the map is just the evaluation map of the sections in the points of Z. Thus $h^1(\mathcal{I}_Z(H)) \geq 1$ if and only if $h^0(\mathcal{I}_Z(H)) \neq 0$, which exactly means that Z is contained in a curve $D \in |\mathcal{O}(H)|$. But Z cannot lie on two different curves $D \neq D'$ since $H^2 = 2 < \mathrm{length}(Z)$ and D and D' cannot have common components because both are reduced and irreducible. $\square$

Let us assume from now on that Z is contained in a curve $D_Z \in |\mathcal{O}(H)|$. We therefore have a section $s \in \mathrm{H}^0(\mathcal{I}_Z(H))$ and by the lemma a unique nontrivial extension

$$0 \to \mathcal{O}_X \xrightarrow{\alpha} \mathcal{F} \to \mathcal{I}_Z(H) \to 0. \tag{28}$$

Lemma 7.6. *Every such non-trivial extension $\mathcal{F}$ is stable.*

Proof. This follows from a more general argument which can be found in [Yosh97, Lem. 2.1]. For the convenience of the reader we shall repeat it in this special case. We only have to check rank one subsheaves. These are all of the form $\mathcal{L} = \mathcal{O}_X(aH) \otimes \mathcal{I}_{Z'}$ with $a \in \mathbb{Z}$ and $Z' \subset X$ a finite length subscheme. The slope of $\mathcal{F}$ is 1, the slope of $\mathcal{L}$ is $2a$. Thus if $\mathcal{L}$ is destabilising, we must have $a \geq 1$. If the induced map $\mathcal{L} \to \mathcal{I}_Z$ is non-zero, we must have $a = 1$. Thus $\mathcal{L} = \mathcal{I}_{Z'}(H)$. The resulting map $\mathrm{Ext}^1(\mathcal{I}_Z(H), \mathcal{O}_X) \to \mathrm{Ext}^1(\mathcal{I}_{Z'}(H), \mathcal{O}_X)$ maps the class of (28) to zero. But the kernel is $\mathrm{Ext}^1(\mathcal{I}_Z(H)/\mathcal{I}_{Z'}(H), \mathcal{O}_X) = 0$. This is a contradiction to the fact that our extension was chosen to be non-trivial. Thus we get a map $\mathcal{L} \to \mathcal{O}_X$, but since $a \geq 1$, this has to be zero, too. $\square$

Proposition 7.7. *Denote the Mukai vector of $\mathcal{F}$ by v_0. The moduli space $\mathcal{M}(v_0)$ is an irrducible symplectic manifold with an induced regular involution ι.*

Proof. Recall that the Picard rank of X is one. Thus the ample cone of X consists of one ray and there are no walls. (This follows since there are no elements in $\mathrm{Pic}\, X$ of negative square (cf. Definition 2.28)). Thus $\mathcal{M}(v_0)$ is an irreducible symplectic manifold. By Proposition 2.35 we have an induced regular involution. $\square$

Remark 7.8. Note that in this special case we can deduce the regularity of the involution ι (as proven in Proposition 2.35) directly: The invariant lattice of $\mathcal{M}(v_0)$ coincides with the Picard group and thus any ample class is mapped to itself. Hence the birational involution ι is regular (cf. [Fuj81, Cor. 3.3]).

Theorem 7.9. *There is a 19-dimensional family of manifolds of $K3^{[2]}$-type admitting a non-symplectic involution with invariant lattice isomorphic to U. Every member of this family is isomorphic to a moduli space of sheaves $\mathcal{M}(2, H, 0)$ on a polarised $K3$ surface (X, H) admitting a double cover to $\mathbb{P}^2$. This family is different from the 19-dimensional family of natural non-symplectic involutions on the Hilbert schemes of two points.*

Proof. Let $\mathcal{K}_1$ denote the moduli space of polarised $K3$ surfaces of degree two. (Every such $K3$ surface is obtained as a smooth double sextic $\pi \colon X \to \mathbb{P}^2$ and the polarisation

H is given by the pullback $\pi^\star\mathcal{O}_{\mathbb{P}^2}(1)$.) Note that $\mathcal{K}_1$ is an irreducible quasi-projective variety. Over $\mathcal{K}_1$ there does not exist a universal family. Following [Sze99, Lem. 2.7], there exists a finite cover $\mathcal{K}' \to \mathcal{K}_1$ with $\mathcal{K}'$ smooth together with a complete family $\psi\colon \mathcal{X} \to \mathcal{K}'$ of degree two $K3$ surfaces. The family $\mathcal{X}$ comes together with a polarisation $\mathcal{L}$ such that for all $t \in \mathcal{K}'$ the restriction $\mathcal{L}_t$ is just given by H, the pullback of a line. As explained in [HL97, Section 6.2], we can construct a relative moduli space of sheaves $\rho\colon \mathcal{M} \to \mathcal{K}'$ on the fibres of ψ such that for every point $t \in \mathcal{K}'$ the fibre $\mathcal{M}_t := \rho^{-1}(t)$ is isomorphic to the moduli space $\mathcal{M}(2, c_1(\mathcal{L}_t), 0)$ of sheaves on the surface $\mathcal{X}_t$. If t corresponds to a surface of Picard rank one, we have seen that the moduli space is smooth. Moreover, the set of points t, where ρ is smooth, is open. Thus we find a Zariski dense open subset of $\mathcal{K}^\circ \subseteq \mathcal{K}'$ such that the restricted family $\mathcal{M}^\circ$ constitues a 19-dimensional family of irreducible symplectic manifolds.

The family $\mathcal{X}$ certainly carries a non-symplectic involution, which preserves the polarisation $\mathcal{L}$. By Proposition 2.35 and Remark 2.36 we see that we have a biregular induced involution on the relative moduli space $\mathcal{M}^\circ \to \mathcal{K}^\circ$ acting fibrewise. If $t \in \mathcal{K}'$ is a point corresponding to a surface $\mathcal{X}_t$ of Picard rank one, then the involution on $\mathcal{M}_t$ is exactly the involution ι discussed above. From Lemma 2.38 we immediately deduce that ι is non-symplectic. The Mukai vector of a sheaf $\mathcal{F}$ in an extension (28) is $v_0 = v(\mathcal{F}) = (2, H, 0)$, so indeed its length is 2 and the invariant lattice is generated by $(1, 0, 0)$ and $(0, H, 1)$. Thus it is isomorphic to U. □

Remark 7.10. The construction (28) of $\mathcal{F}$ is a so-called Serre-construction, a correspondence between codimension two subschemes and rank two vector bundles as a generalisation to the correspondence between divisors and line bundles. The degeneracy locus of global sections of $\mathcal{F}$ define codimension two subschemes of X, the defining section α in (28) corresponds to the subscheme Z itself.

Remark 7.11. Note that the general involutions in the families No. 2 and 3 of Theorem 7.3 cannot be realised as moduli spaces of sheaves whose involutions are naturally induced from an involution of the surface (cf. [OW13]).

7.3 The Fixed Locus

We continue with a few sheaf-theoretic considerations in order to understand the fixed locus F of the involution ι on the moduli space $\mathcal{M}(v_0)$.

Lemma 7.12. *We have $h^0(\mathcal{F}) = 2$.*

Proof. This follows easily from (28) since $\mathrm{H}^0(\mathcal{I}_Z(H)) = \mathbb{C}s$. □

Lemma 7.13. *Two length three subschemes $Z, Z' \in \mathcal{B}$ define isomorphic sheaves $\mathcal{F}$ if and only if $D_Z = D_{Z'}$ and $\mathcal{O}_{D_Z}(-Z) \simeq \mathcal{O}_{D_{Z'}}(-Z')$.*

Proof. The "if" direction is trivial. Conversely, let s and s' be the sections of $\mathcal{O}_X(H)$ vanishing on Z and Z', respectively. They yield injections $\mathcal{O}_X \xrightarrow{s} \mathcal{I}_Z$ and $\mathcal{O}_X \xrightarrow{s'} \mathcal{I}'_Z$. Furthermore, for Z and Z' we have unique extensions of the form (28) (p.74) and by assumption the two sheaves obtained are isomorphic. Altogether we have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_X \arrow[r, equal] \arrow[d, "\alpha'"] & \mathcal{O}_X \arrow[d, "s"] & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\alpha"] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{I}_Z(H) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "s'"] & \mathcal{I}_{Z'}(H) \arrow[r] \arrow[d] & \mathcal{Q} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd},$$

where the quotient $\mathcal{Q}$ is $\mathcal{O}_{D_Z}(H|_{D_Z} - Z) \simeq \mathcal{O}_{D_{Z'}}(H|_{D_{Z'}} - Z')$. □

Let $D \in |\mathcal{O}(H)|$ be a smooth curve and $Z \subset D$ a length three subscheme. We consider the degree three line bundle $\mathcal{O}_D(Z)$. Using Grothendieck−Riemann−Roch we can easily compute its Mukai vector:

$$v' := v(\mathcal{O}_D(Z)) = (0, H, 2).$$

Lemma 7.14. *We have $\mathcal{M}(v_0) \cong \mathcal{M}(v')$.*

Proof. Note that by the same argument as in the proof of Proposition 7.7 the moduli space $\mathcal{M}(v')$ is an irreducible symplectic fourfold. We can make the isomorphism explicit (this was pointed out by M. Lehn): For a sheaf $\mathcal{F} \in \mathcal{M}(v_0)$ we consider the following short exact sequence:

$$0 \to \mathcal{O}_X \otimes \mathrm{H}^0(X, \mathcal{F}) \xrightarrow{\mathrm{ev}} \mathcal{F} \to \mathcal{Q} \to 0.$$

By Lemma 7.12 the quotient $\mathcal{Q}$ is a torsion sheaf supported on a curve $D \in |\mathcal{O}(H)|$, which coincides with $\mathcal{Q} \simeq \mathcal{O}_D(H|_D - Z)$ from the proof of Lemma 7.13 above for some choice of a length three subscheme $Z \subset D$ defined by a section of $\mathcal{F}$. Dualising $\mathcal{Q}$ and then tensoring with $\mathcal{O}_D(H|_D)$ gives the corresponding point in $\mathcal{M}(v')$. By Lemma 7.13 this is an isomorphism. □

Remark 7.15. The moduli space $\mathcal{M}(v')$ can be regarded as a relative compatified Jacobian: Denote by $\mathcal{U} \to |\mathcal{O}(H)|$ the universal family of curves in $|\mathcal{O}(H)|$ (cf. [HL97, Sect. 3.1]). Inside $X^{[3]}$ we have the relative Hilbert scheme $\mathcal{U}^{[3]} \to |\mathcal{O}(H)|$ of subschemes

$Z \in X^{[3]}$ lying on a curve $D_Z \in |\mathcal{O}(H)|$. There is a relative compactified Abel map

$$f\colon \mathcal{U}^{[3]} \to \mathcal{M}(v')$$

over $|\mathcal{O}(H)|$ with generic fibre a projective line, which — at least over points corresponding to smooth curves — is given by the assignment

$$\mathcal{U}^{[3]} \ni Z \mapsto \mathcal{O}_{D_Z}(Z).$$

If D is a smooth curve, the fibre over D of the map

$$g\colon \mathcal{M}(v') \to |(\mathcal{O}(H)|, \quad \mathcal{O}_D(Z) \mapsto D$$

is precisely the Jacobian $\mathcal{J}^3D$ of degree three line bundles on D.

By Lemma 7.12 each sheaf $\mathcal{F}$ defines a one dimensional family of length three subschemes in X, which exactly corresponds to the fibre of f over $\mathcal{F}$, where we regard $\mathcal{F}$ as a point in $\mathcal{M}(v')$ via the isomorphism $\mathcal{M}(v_0) \cong \mathcal{M}(v')$ of Lemma 7.14.

Certainly $\mathcal{U}^{[3]}$ is invariant under the induced involution $\varphi^{[3]}$ on $X^{[3]}$ and f is equivariant with respect to $\varphi^{[3]}$ and the action ι on $\mathcal{M}(v_0) \cong \mathcal{M}(v')$. Hence $g\colon \mathcal{M}(v') \to |\mathcal{O}(H)|$ is a ι-invariant Lagrangian fibration.

Theorem 7.16. *The fixed locus of* $(\mathcal{M}(v_0), \iota)$ *consists of two smooth connected surfaces* F_1 *and* F_2 *which are both branched coverings of* $\mathbb{P}^2$ *of degree six and ten, respectively.*

Proof. We use the isomorphism $\mathcal{M}(v_0) \cong \mathcal{M}(v')$ of Lemma 7.14. By Remark 7.15 we see that we have a ι-invariant fibration $g\colon \mathcal{M}(v_0) \to |\mathcal{O}(H)|$, where the action on the base is, of course, trivial. Therefore φ acts on every fibre. A general point $D \in |\mathcal{O}(H)|$ corresponds to a smooth genus two curve which is a double cover of a line ramified at six points $p_1, \dots, p_6$ which are exactly the points in $D \cap C$. Also they are the fixed points of the hyperelliptic involution ι_D on D. The fibre $g^{-1}(D)$ can be identified with the Jacobian $\mathcal{J}^3D$ and the involution on $\mathcal{J}^3D$ is given by pulling back divisors along ι_D. There are exactly 16 fixed points in $\mathcal{J}^3D$ which are all of the form $p_i + p_j + p_k$ for some i, j, k. We divide the set of fixed points into two sets. The first consists of the six classes of divisors of the form $3p_i$ for some i. The second set consists of classes of divisors of the form $p_i + p_j + p_k$ for i, j, k distinct. Now let $C^\star \subset |\mathcal{O}(H)|$ denote the locus of tangent lines to C, i.e. the dual curve. As the curve D moves in the open set $|\mathcal{O}(H)| \setminus C^\star$ the fixed points deform with it in the obvious way. In this way we obtain two surfaces $\tilde{F}_1$ and $\tilde{F}_2$ which are unramified coverings of $|\mathcal{O}(H)| \setminus C^\star$ of degree six and ten.

What is left to do, is to show that the closures F_1 and F_2 of $\tilde{F}_1$ and $\tilde{F}_2$, respectively, do not intersect and are both connected. (Here I want to thank Manfred Lehn for pointing out an error in an earlier version of this theorem and for indicating the beautiful proof of the revised statement.) For the first assertion we define a function on the set of degree three line bundles $\mathcal{L}$ on curves $D \in |\mathcal{O}(H)| \setminus C^\star$ as follows: For any such curve D and any line bundle $\mathcal{L}$ on D we consider the space of global sections $\mathrm{H}^0(D, \mathcal{L})$. The hyperelliptic involution ι_D acts by pullback on this vector space. We set $r(\mathcal{L}) := \dim \mathrm{H}^0(D, \mathcal{L})^{\iota_D}$.

For a line bundle $\mathcal{L}_1$ corresponding to a divisor $3p_i$ we have $r(\mathcal{L}_1) = 2$: Every divisor in the linear system $|\mathcal{L}_1|$ is of the form $p_i + p + \iota_D(p)$ for some $p \in D$. This gives a two-dimensional space of sections which are all ι_D-invariant. On the other hand, for a line bundle $\mathcal{L}_2$ associated with a divisor $p_i+p_j+p_k$ with i, j, k distinct, the only fixed divisors in this linear system are $p_i + p_j + p_k$ and $p_l + p_m + p_n$ with $\{i, j, k, l, m, n\} = \{1, \ldots, 6\}$. Thus $r(\mathcal{L}_2) = 1$. This analysis shows that the function r takes different values on open parts of the surfaces F_1 and F_2. Since the fixed locus is smooth, the two surfaces cannot intersect.

In order to show that F_1 and F_2 are connected, it is certainly enough to show that we can deform a divisor of the form $p_1 + p_j + p_k$ on a smooth genus two curve D into the divisor $p_2 + p_j + p_k$. We consider the situation that D specialises to a nodal elliptic curve, where the points p_1 and p_2 come together. (This is the double cover picture of a general line in $\mathbb{P}^2$ specialising to a tangent.) Denote the node by q. The limits of the divisors p_1 and p_2 in the compactified Jacobian are both given by the divisor q. □

Remark 7.17. In order to proof that the surfaces F_1 and F_2 do not intersect one can also proceed as follows: We want to show that we cannot deform a divisor of the form $2p_1$ to the divisor $p_1 + p_2$. Again, we can look at the limits of these divisors in the compactified Jacobian of the singular curve, where p_1 and p_2 come together to form a node q. The support of both limits consists of the node q, but the scheme structure is different. The limit of the divisor $p_1 + p_2$ is a length two subscheme supported at q consisting of the point q together with a *horizontal* tangent vector. (Here we think of all curves being branched over a horizontal $\mathbb{P}^1$.) On the other hand, the divisor $2p_1$ is a fibre of the two-to-one map to $\mathbb{P}^1$. Thus the limit point has to be a fibre, too. Indeed, the limit is a length two subscheme consisting of the node q together with a *vertical* vector. This length two subscheme collapses when mapped to $\mathbb{P}^1$.

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