Regularity of a Degenerate Oblique Derivative Problem

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Abstract

The purpose of this thesis is to prove an existence and uniqueness theorem for a degenerate oblique derivative problem associated with a second order uniformly strongly elliptic differential operator with coefficients of limited regularity in the framework of Sobolev spaces.
Specifically, on a compact manifold $\overline{D}$ we study the resolvent to the boundary value problem

$$\begin{cases} Au = f & \text{in } D \\ Lu = \mu_1(x')\frac{\partial u}{\partial n}(x') + \mu_2(x')u(x') = \varphi & \text{on } \partial D \end{cases}$$

for a uniformly strongly elliptic second order operator A with C^τ coefficients and C^τ functions μ_1, μ_2 with $\mu_1, \mu_2 \geq 0$, $\mu_1(x') + \mu_2(x') > 0$, $\tau \geq 4$.

Important tools are the calculus of pseudodifferential operators with non-smooth coefficients and the resolvent construction for the Dirichlet and Neumann problems in the case of non-smooth coefficients. The solvability of the degenerate boundary value problem requires the invertibility of the hypoelliptic operator $T = \mu_1\Lambda + \mu_2$ on the boundary. Here Λ is the Dirichlet-to-Neumann operator. In order to establish the invertibility, we show that for $\tilde{\mu} \geq 0$ the operators $T - \tilde{\mu}$ form a Fredholm family of index zero and that T is injective.

In the first part the mapping properties and compositions of operator-valued pseudodifferential operators with Hölder continuous coefficients and transmission condition for C^τ_* symbols are treated, and the parametrix of hypoelliptic operators is analyzed. In the second part, we generalize the resolvent construction for the Dirichlet problem in the non-smooth situation to all $\tau > 0$. In the third part, it is proven that the above boundary value problem for second order differential operators with C^τ coefficients is solvable for $\tau \geq 4$.

Keywords:

Degenerate boundary value problem; pseudodifferential operators with Hölder continuous coefficients; transmission condition for C^τ_* symbols; hypoellipticity.

Zusammenfassung

Ziel dieser Arbeit ist der Beweis eines Existenz- und Eindeutigkeitssatzes für ein entartes Randwertproblem mit schräger Ableitung zu einem gleichmäßig elliptischen Differentialoperator zweiter Ordnung mit Koeffizienten eingeschränkter Regularität mittels Sobolevraummethoden.

Konkret betrachen wir eine kompakte Mannigfaltigkeit $\overline{D}$ die Resolvente des Randwertproblems

$$\begin{cases} Au = f & \text{in } D \\ Lu = \mu_1(x')\frac{\partial u}{\partial n}(x') + \mu_2(x')u(x') = \varphi & \text{auf } \partial D \end{cases}$$

für einen gleichmäßig elliptischen Operator A zweiter Ordnung mit Koeffizienten in C^τ und C^τ-Funktionen μ_1, μ_2 mit $\mu_1, \mu_2 \geq 0$, $\mu_1(x') + \mu_2(x') > 0$ und $\tau \geq 4$.

Wichtige Hilfsmittel sind der Kalkül für Pseudodifferentialoperatoren mit nicht-glatten Koeffizienten und die Konstruktion der Resolvente für das Dirichlet- und das Neumannproblem im Fall nicht-glatter Koeffizienten. Die Lösbarkeit des entarteten Randwertproblems mit schräger Ableitung erfordert die Invertierbarkeit des hypoelliptischen Operators $T(\lambda) = \mu_1\Lambda + \mu_2$ auf dem Rand. Dabei ist Λ der Dirichlet-Neumann Operator. Zum Nachweis der Invertierbarkeit wird gezeigt, dass für $\tilde{\mu} \geq 0$ die $T(\lambda) - \tilde{\mu}$ eine Fredholmfamilie vom Index Null bilden und dass $T(\lambda)$ injektiv ist.

Im ersten Teil werden Abbildungseigenschaften und Verknüpfungen von operatorwertigen Pseudodifferentialoperatoren mit hölderstetigen Koeffizienten und Transmissionsbedingung für C^τ-Symbole behandelt und die Parametrix hypoelliptischer Operatoren analysiert. Im zweiten Teil verallgemeinern wir die Konstruktion der Resolvente für das Dirichletproblem mit nicht glatten Koeffizienten von beliebiger Regularität $\tau > 0$. Im dritten Teil wird gezeigt, dass das obige entartete Randwertproblem für Differentialoperatoren zweiter Ordnung mit C^τ-Koeffizienten lösbar ist, falls $\tau \geq 4$.

Schlüsselworte:

Entartete Randwertaufgabe, Pseudodifferentialoperatoren mit hölderstetigen Koeffizienten, Transmissionsbedingung für C^τ-Symbole, Hypoelliptizität

Contents

Chapter 1

Introduction

In this thesis we shall prove a unique solvability theorem for a degenerate oblique derivative problem with real coefficients. The background is some work of Taira [18], [19], [20], [21] and Kannai [13].

Let $\overline{D} = D \cup \partial D$ be an n-dimensional, compact C^∞ manifold with boundary. Let

$$A(x, D) = \sum_{i,j=1}^{n} a^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b^i(x)\partial_{x_i} + c(x)$$

be a second-order elliptic differential operator with real coefficients. For fixed $\tau \geq 4$ we assume the following conditions:

A.1) $a^{ij} \in C^\tau(\overline{D})$ (Definition 2.1.1), $a^{ij} = a^{ji}$ and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \,, x \in \overline{D} \,, \xi \in \mathbb{R}^n.$$

A.2) $b^i \in C^\tau(\overline{D})$.

A.3) $c \in C^\tau(\overline{D})$ and $c \leq 0$ on $\overline{D}$.

We consider the following boundary value problem: Given functions f and φ defined in D and ∂D respectively, find a function u in D such that

$$\begin{cases} (A - \lambda)u = f & \text{in } D \\ Lu = \mu_1(x')\frac{\partial u}{\partial n}(x') + \mu_2(x')u(x') = \varphi & \text{on } \partial D. \end{cases} \tag{1.1}$$

Here:
1) λ is a complex number in the sector

$$\left\{\lambda = r^2 e^{i\eta}, -\epsilon + \pi < \eta < \pi - \epsilon\right\} \tag{1.2}$$

for some $0 < \epsilon < \pi$.
2) $\mu_1 \in C^\tau(\partial D)$ and $\mu_1 \geq 0$ on ∂D.

3) $\mu_2 \in C^\tau(\partial D)$ and $\mu_2 \geq 0$ on ∂D.
4) $\frac{\partial}{\partial n}$ is the normal derivative.

Our fundamental hypothesis is the following :

$$\mu_1(x') + \mu_2(x') > 0,\ x' \in \partial D\ . \tag{1.3}$$

Degenerate boundary conditions of this form arise from stochastic diffusion processes where particles are either reflected or absorbed at the boundary. It is well known that an efficient way of studying boundary value problems is to transform them into problems involving pseudodifferential operators on the boundary. This method was used by Hörmander [10] to study hypoellipic boundary problems.

Problem (1.1) was considered by Taira [20] in the case of smooth coefficients and domains. We generalize certain key results from his studies to the non-smooth case.

We consider $(A - \lambda, L)$ as an unbounded operator in $H^s(\overline{D}) \times H^{s+3/2}(\partial D)$ (Sobolev spaces) with domain $H_L^{s+2} = \left\{u \in H^{s+2}; Lu \in H^{s+\frac{3}{2}}\right\}$. It is easy to see that $(A - \lambda, L)$ is closed and densely defined.

We obtain Theorem 4.0.4:

Under the above assumptions on A and L, the mapping

$$(A - \lambda, L) : H_L^{s+2}(\overline{D}) \to H^s(\overline{D}) \times H^{s+3/2}(\partial D)$$

is a topological isomorphism for all

$$0 \leq s < \tau - 3,$$

for all λ in (1.2), $|\lambda|$ sufficiently large.

The structure of the present thesis is as follows:

Studies of pseudodifferential operators whose symbols $p(x, \xi)$ satisfy a Hölder and Zymund conditions in x have been found to be very useful in PDE. A number of their properties and applications have been investigated by Marschall [14], [15] and Taylor [22], and in other places. In **Chapter 2** we introduce two types of function spaces. First, Hölder space $C^\tau(\mathbb{R}^n)$ for $\tau \in \mathbb{R}_{>0}$. Second, Zygmund space $C_*^\tau(\mathbb{R}^n)$ for $\tau \in \mathbb{R}$. We discuss the elementary properties of operators with symbols in $C^\tau S^m_{1,\delta}$ and $C_*^\tau S^m_{1,\delta}$ for $0 \leq \delta < 1$. Also the decomposition of a symbol $p \in C_*^\tau S^m_{1,\delta}$ into a sum of two terms is established. One, $p^\sharp(x, \xi)$, has better smoothness properties, of use in further results on operator calculus. The other, $p^b(x, \xi)$, has no better smoothness, but does have a lower order (by a degree depending on the smoothness of $p(x, \xi)$).

We consider the composition of pseudodifferential operators with non-smooth coefficients. We will show under certain restrictions, that if $p_1 \in C_*^{\tau_1} S^{m_1}_{1,\delta_1}$ and $p_2 \in C_*^{\tau_2} S^{m_2}_{1,\delta_2}$, for any $0 < \theta < \tau_2$, $\theta \notin \mathbb{N}$,

$$p_1(x, D_x)p_2(x, D_x) = \sum_{|\alpha| \leq [\theta]} \frac{1}{\alpha!} Op(\partial_\xi^\alpha p_1(x, \xi) D_x^\alpha p_2(x, \xi)) + \mathcal{R}(x, D_x),$$

where $\mathcal{R}$ is of order $m_1 + m_2 - (1 - \delta_2)\theta$ in the sense of mapping properties in Sobolev space (Theorem 2.1.18). In [1, Theorem 3.6], Abels proved a similar theorem in the context of non-smooth symbols of the class $C^\tau_* S^m_{1,0}$, and in [15, Theorem 3], Marschall stated a similar theorem without proof.

Then we recall the transmission condition for non-smooth pseudodifferential operators by Abels [1]. Abels investigated the effect of the transmission condition for non-smooth truncated pseudodifferential operators $r^+ p(x, D_x)(u \otimes \delta)$ (Lemma 2.2.1). In [10, Theorem 2.1.4], Hörmander studied boundary values on a surface of the potential of a multiple layer with respect to a pseudodifferential operator with smooth coefficients. In fact, he showed the following:

Let Q be a pseudodifferential operator in $D \subset \mathbb{R}^n$ such that every term in the asymptotic expansion of the symbol $\sum q_j(x, \xi)$ is a rational function of ξ. Then for every $u \in C^\infty_0(\partial D)$ the boundary values

$$Q^{lm} u = \lim_{x_n \to +0} D^l_n Q(u \otimes \delta^m),$$

where $\delta^m = D^m_n \delta(x_n)$, are given by pseudodifferential operators Q^{lm} with symbols

$$q_{lm}(x', \xi') = \sum_j (2\pi)^{-1} \int_{\Gamma_{\xi'}} (D_n + \xi_n)^l q^j(x', 0, \xi) \xi^m_n d\xi_n. \tag{1.4}$$

Here $\Gamma_{\xi'}$ is a contour in $\mathbb{C}$, which has the poles of q^j in the upper half plane in its interior.

Next we combine the results of Hörmander and Abels to show Hörmander's formula for non-smooth symbols with transmission property (Theorem 2.4.2). These explicit expressions are then used to compute the terms in the symbol expansion of certain pseudodifferential operators on the boundary. Further, we introduce a parameter-dependent hypoellipticity condition for non-smooth symbols and construct a parametrix for a pseudodifferential operator with symbol in $C^\tau_* S^m_{1,\delta}$ for $0 \leq \delta < 1$. In the smooth case, a similar parametrix was constructed by Bilyi, Schrohe, Seiler [5] to discuss the H_∞-calculus of hypoelliptic operators. A basic theorem says:

Let $p \in C^\tau_* S^m_{1,\delta}$ be hypoelliptic in the sense of Definition 2.5.1 and let $0 \leq \delta \leq \delta' < 1$. Consider the following sector:

$$S(\epsilon) = \left\{ \lambda = r^2 e^{i\eta}, \epsilon \leq \eta \leq 2\pi - \epsilon \right\}$$

for some $0 < \epsilon < \pi$. Then for each $\theta \notin \mathbb{N}$, $\theta < \tau$ and $0 < \theta < \tau - [\theta]$, there exist right and left parametrices $q^r(x, \xi, \lambda), q^l(x, \xi, \lambda) \in C^{\tau - [\theta]}_* S^0_{1,\delta'}$ such that the operators

$$\mathcal{R}^r(x, D_x, \lambda) = (p(x, D_x) - \lambda) q^r(x, D_x, \lambda) - I : H^{s+m-\theta(1-\delta')}(\mathbb{R}^n) \to H^s(\mathbb{R}^n),$$

for

$$\begin{cases} -(\tau - [\theta])(1 - \delta) & < \quad s < \tau - [\theta], \\ -(1 - \delta')(\tau - [\theta] - \theta) & < \quad s + m < \tau - [\theta], \\ -(1 - \delta')(\tau - [\theta] - \theta) & < \quad s \end{cases}$$

and

$$\mathcal{R}^l(x,D_x,\lambda) = q^l(x,D_x,\lambda)(p(x,D_x)-\lambda) - I : H^{s+m-\theta(1-\delta')}(\mathbb{R}^n) \to H^s(\mathbb{R}^n),$$

for

$$\begin{cases} -(\tau-[\theta])(1-\delta') & < \quad s < \tau - [\theta], \\ -(1-\delta)(\tau-\theta) & < \quad s < \tau, \end{cases}$$

are bounded. The norms of $\mathcal{R}^r(x,D_x,\lambda)$ and $\mathcal{R}^l(x,D_x,\lambda)$ are $O(\langle\lambda\rangle^{-1})$ in $S(\epsilon)$.

The resolvent of the Dirichlet problem $\begin{pmatrix} A-\lambda \\ \gamma_0 \end{pmatrix}$ with C^τ-coefficients for $0 < \tau \leq 1$ has been constructed by Abels, Grubb and Wood in [2, Theorem 4.1]. In **Chapter 3** we use their ideas to show a similar results for the Dirichlet problem and the Neumann problem $\begin{pmatrix} A-\lambda \\ \gamma_1 \end{pmatrix}$ with C^τ-coefficients for all $\tau > 0$.

Using the results of Chapter 3 we reduce in **Chapter 4** the proof of the unique solvability of (1.1) to the invertibility of the first order pseudodifferential operator

$$T(\lambda) = \mu_1\Lambda(\lambda) + \mu_2$$

on the boundary ∂D, where $\Lambda(\lambda)$ is a Dirichlet-to-Neumann type operator and

$$D(T(\lambda)) = \left\{\psi \in H^{s+\frac{3}{2}}(\partial D); T(\lambda)\psi \in H^{s+\frac{3}{2}}(\partial D)\right\}$$

(Lemma 4.0.5). We explain here in three steps the proof of the invertibility of $T(\lambda)$.

Step 1. We show that $T(\lambda)$ is a Fredholm operator if and only if $Op(t_1+t_0)$ is a fredholm operator.
We apply Hörmander's formula (Theorem 2.4.2) to compute the first two terms in the symbol expansion of the Calderón projector and the Dirichlet-to-Neumann operator in the non-smooth situation (Lemma 4.1.1 and 4.2.1). Then we determine the first two terms in the asymptotic expansion of $T(\lambda)$ - we denote them by t_1 and t_0- with the help of the first two terms in the asymptotic expansion of the Dirichlet-to-Neumann operator $\Lambda(\lambda)$. Since we can write

$$T(\lambda) = Op(t_1+t_0) + \overbrace{Op(t_{-1}+t_{-2}+\cdots)}^{\text{compact operator}}, \tag{1.5}$$

we focus just on t_1+t_0. Then

$$\text{ind } T(\lambda) = \text{ind } Op(t_1+t_0).$$

Step 2. We prove that ind $Op(t_1+t_0) = 0$.
First we show that t_1+t_0 is hypoelliptic in the sense of Definition 2.5.1. Then we use the basic theorem on hypoellipticity to construct a parameter-dependent parametrix $Op(b_0) \in C^{\tau-1}S^0_{1,\frac{1}{2}}$ for $t_1+t_0-\tilde{\mu} \in C^{\tau-1}S^1_{1,0}$ for some fixed $\tilde{\lambda}$ in the sector (1.2), where $\tilde{\mu}$ lies in $\{|\tilde{\mu}| < c\}$ or outside a suitable sector around the

positive real axis. As a result, $Op(t_1 + t_0 - \tilde{\mu})$ is invertible for large $|\tilde{\mu}|$ and hence is a Fredholm operator with index zero for all $\tilde{\mu}$. By (1.5) $T(\lambda)$ is a Fredholm operator with zero index for all λ in the Sector (1.2). Since $T(\lambda)$ is a Fredholm operator, we obtain the following a priori estimate

$$\|\psi\|_{H^{s+\frac{3}{2}}(\partial D)} \leq C'\Big(\|T(\lambda)\psi\|_{H^{s+\frac{3}{2}}(\partial D)} + \|\psi\|_{H^{s-\frac{1}{2}}(\partial D)}\Big), \quad \psi \in D(T(\lambda)). \tag{1.6}$$

Step 3. We prove that $T(\lambda)$ is injective.
We introduce a linear operator $A_L : D(A_L) \subset H^{s+2}(D) \to H^s(D)$ with

$$D(A_L) = \Big\{u \in H^{s+2}(D); Lu = 0\Big\}$$

and prove the following fundamental a priori estimate for $A_L - \lambda$

$$\|u\|^2_{H^{s+2}} + \langle r\rangle^{s+2}\|u\|^2_{L^2} \leq C''(\eta)\Big(\|(A_L - \lambda)u\|^2_{H^{s+2}} + \langle r\rangle^s\|(A_L - \lambda)u\|^2_{L^2}\Big),$$

with a constant $C''(\eta) > 0$, $|\lambda|$ sufficiently large, and for

$$0 \leq s < \tau - 3.$$

This shows that A_L is injective (Theorem 4.3.1). In the proof of this theorem we make use of Agmon's method [3]. This is a technique of treating a spectral parameter λ as a second order differential operator of an extra variable and relating the old problem to a new one with additional variable. Then we show that the a priori estimate (1.6) for the operator $T(\lambda)$ is equivalent to the a priori estimate

$$\|u\|_{H^{s+2}(D)} \leq C'\Big(\|(A_L - \lambda)u\|_{H^s(D)} + \|u\|_{H^s(D)}\Big), \ u \in D(A_L).$$

Finally, since

$$N\Big((A - \lambda, L)\Big) = \Big\{u \in H^{s+2}_L; (A - \lambda, L)u = 0\Big\} = N(A_L - \lambda),$$

we conclude that $N\Big(T(\lambda)\Big) = 0$. Therefore, $T(\lambda)$ is injective and invertible.

Chapter 5 contains details on the resolvent construction for the Dirichlet problem and details on the calculation of the second term in the asymptotic expansion of the symbol $C^+_{\lambda,00}$ in the Calderón projector.

Chapter 2

Preliminaries

2.1 Pseudodifferential Operators with Non-Smooth Coefficients and Poisson Operators

In this section we recall the definitions and properties of function spaces that will be used throughout this thesis and the mapping properties of non-smooth pseudodifferential operators. We apply the technique outlined by Abels in [1], extending the composition of non-smooth pseudodifferential operators from the case $\delta = 0$ to all $0 \leq \delta < 1$, which will be the basis for the further discussion.

In the sequel let X, X_0 and X_1 be Banach spaces.

Definition 2.1.1. *For $\tau \in (0, \infty)$, the Hölder space $C^\tau(\mathbb{R}^n, X)$ is the set of all functions $f : \mathbb{R}^n \to X$ such that*

$$\| f \|_{C^\tau(\mathbb{R}^n;X)} := \sum_{|\alpha| \leq [\tau]} \| \partial_x^\alpha f \|_{L^\infty(\mathbb{R}^n;X)} + \sum_{|\alpha|=[\tau]} \sup_{x \neq y} \frac{\|\partial_x^\alpha f(x) - \partial_x^\alpha f(y)\|_X}{| x - y |^{\tau - [\tau]}} < \infty.$$

Cf. [1, p.1471]. *We write $C^\tau(\mathbb{R}^n)$ for $C^\tau(\mathbb{R}^n, \mathbb{C})$.*

We will use a partition of unity

$$1 = \sum_{h=0}^{\infty} \varphi_h, \quad \varphi_h \text{ supported on } \langle \xi \rangle \sim 2^h$$

such that $\varphi_h(\xi) = \varphi_1(2^{1-h}\xi)$ for $h \geq 2$. To get this you can start with non-negative $\varphi_0(\xi)$, equal to 1 for $|\xi| \leq 1$, 0 for $|\xi| \geq 2$, set $\phi_h(\xi) = \varphi_0(2^{-h}\xi)$, and set $\varphi_h(\xi) = \phi_h(\xi) - \phi_{h-1}(\xi)$ for $h \geq 1$. We will call this a Littlewood-Paley partition of unity.

Proposition 2.1.2. *For any $\xi \in \mathbb{R}^n$, $\alpha, \gamma \in \mathbb{Z}_+^n$ and some positive constant $C_{\alpha,\gamma}$:*

$$\sum_{h=0}^{\infty} \varphi_h(\xi) = 1; \tag{2.1}$$

$$\sum_{h=0}^{\infty} \varphi_h(D)u = u, \text{ with convergence in } S'(\mathbb{R}^n); \tag{2.2}$$

$$|\xi^\gamma \partial^{\alpha+\gamma} \varphi_h(\xi)| \leq C_{\alpha,\gamma} 2^{-|\alpha|h}, \; h = 0, 1, ... \tag{2.3}$$

Proof. Cf. [22, Chapter 13, Section 9] or [6, Proposition 2.4]. □

Definition 2.1.3. *Let $\tau \in \mathbb{R}$. We define the Zygmund space $C_*^\tau(\mathbb{R}^n)$ to consist of all u such that*

$$\|u\|_{C_*^\tau} = \sup_h 2^{h\tau} \|\varphi_h(D)u\|_{L^\infty} < \infty.$$

Cf. [22, p.38].

Remark 2.1.4.

$$\begin{cases} C^\tau = C_*^\tau, & \text{if } \tau \in \mathbb{R}_{>0} \setminus \mathbb{Z}_{>0} \\ C^\tau \subset C_*^\tau, & \text{if } \tau \in \mathbb{Z}_{\geq 0} \end{cases}.$$

Cf. [22, p.38].

Lemma 2.1.5. *If $\tau \geq 1$ and $f, g \in C^\tau$, then $f \circ g \in C^\tau$ and*

$$\|f \circ g\|_{C^\tau} \leq C\Big(\|f\|_{C^\tau}\|g\|_{C^1}^\tau + \|f\|_{C^1}\|g\|_{C^\tau} + \|f\|_{C^0}\Big). \tag{2.4}$$

Proof. In more general form, this estimate can be found in [11, Theorem A.8]. □

Definition 2.1.6. *a) The space $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ of symbols of order m, for $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$ is defined as the set of all functions $p(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0^n$, there is a constant $C_{\alpha,\beta}$ such that*

$$|D_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}.$$

Here $\langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}}$. Cf. [9, Chapter 7, Section 7.1].

b) Let $0 \leq \delta \leq 1$, $\tau > 0$ and $m \in \mathbb{R}$. The space $C_*^\tau S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; X)$ is the set of all functions $p : \mathbb{R}^n \times \mathbb{R}^n \to X$ that are smooth with respect to ξ and are C_*^τ with respect to x satisfying the estimates

$$\|D_\xi^\alpha \partial_x^\beta p(.,\xi)\|_{L^\infty(\mathbb{R}^n;X)} \leq C_{\alpha,\beta}\langle \xi \rangle^{m-|\alpha|+\delta|\beta|}, \ \|D_\xi^\alpha p(.,\xi)\|_{C_*^\tau(\mathbb{R}^n;X)} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|+\delta\tau}$$

for all $\alpha \in \mathbb{N}_0^n$ and $|\beta| \leq [\tau]$. Cf. [1, Definition 3.1].

c) Let $0 \leq \delta \leq 1$, $\tau \geq 0$ and $m \in \mathbb{R}$. The space $C^\tau S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; X)$ is the set of all functions $p : \mathbb{R}^n \times \mathbb{R}^n \to X$ that are smooth with respect to ξ and are C^τ with respect to x satisfying the estimates

$$\begin{aligned} \|D_\xi^\alpha p(.,\xi)\|_{C^{|\beta|}(\mathbb{R}^n;X)} &\leq C_{\alpha,\beta}\langle \xi \rangle^{m-|\alpha|+\delta|\beta|}, \ 0 \leq |\beta| \leq \tau \\ \|D_\xi^\alpha p(.,\xi)\|_{C^\tau(\mathbb{R}^n;X)} &\leq C_\alpha \langle \xi \rangle^{m-|\alpha|+\delta\tau} \end{aligned}$$

for all $\alpha \in \mathbb{N}_0^n$.

Definition 2.1.7. *For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^n)$ the Sobolev space*

$$H^s(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n); \langle \xi \rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}.$$

It is a Hilbert space with the following scalar product and norm

$$(u,v)_s = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}\langle \xi \rangle^{2s} d\xi,$$

$$\|u\|_s = \|\langle\xi\rangle^s \hat{u}(\xi)\|_{L^2}.$$

Clearly

$$\mathcal{F} : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^2, \langle\xi\rangle^{2s} d\xi)$$

is an isometry, where $\mathcal{F}$ is the Fourier transformation.

We define the pseudodifferential operator $p(x, D_x) = Op(p)$ associated to a symbol $p \in C^\tau_* S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0, X_1))$, $0 \le \delta < 1$, by

$$p(x, D_x)u(x) = Op(p)u(x) = \int_{\mathbb{R}^n} e^{ix\xi} p(x,\xi)\hat{u}(\xi) đ\xi\ ,\ u \in S(\mathbb{R}^n; X_0),$$

where $đ\xi := (2\pi)^{-n} d\xi$. Write $x = (x', x_n)$, $\xi = (\xi', \xi_n)$. For fixed x', ξ' define

$$Op_n(p)u(x) = \int_{\mathbb{R}} e^{ix_n\xi_n} p(x', x_n, \xi', \xi_n)\hat{u}(\xi_n) đ\xi_n\ ,\ u \in S(\mathbb{R}^n; X_0).$$

Remark 2.1.8. *a) Let $p_1 \in C^\tau S^{m_1}_{1,\delta_1}$ and $p_2 \in C^\tau S^{m_2}_{1,\delta_2}$ for $m_1, m_2 \in \mathbb{R}$, $0 \le \delta_1, \delta_2 < 1$ and $\tau > 0$. Then from the well-known inequality for Hölder norms of products,*

$$\|uv\|_{C^\tau} \le C\Big(\|u\|_{C^\tau}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{C^\tau}\Big)\ ,\ u, v \in C^\tau, \tag{2.5}$$

we have

$$p_1 p_2 \in C^\tau S^{m_1+m_2}_{1,\delta},$$

where $\delta = \max\{\delta_1, \delta_2\}$. We can easily extend the inequality (2.5) *to the case of n functions, i.e. if $u_i \in C^\tau$ for $1 < i \le n$, then*

$$\|u_1 u_2 ... u_n\|_{C^\tau} \le \sum_{j=1}^{n} \|u_j\|_{C^\tau}\Big(\Pi^n_{\substack{i=1\\ i\neq j}} \|u_i\|_{L^\infty}\Big).$$

In more general form, this estimate can be found in [14, p.944].

b) Let $p_1 \in C^\tau_* S^{m_1}_{1,\delta_1}$ and $p_2 \in C^\tau_* S^{m_2}_{1,\delta_2}$ for $m_1, m_2 \in \mathbb{R}$ and $0 \le \delta_1, \delta_2 < 1$. Then from the inequality

$$\|uv\|_{C^\tau_*} \le C\Big(\|u\|_{C^\tau_*}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{C^\tau_*}\Big)\ ,\ u, v \in C^\tau_*, \tag{2.6}$$

[4, Proposition 2.1.1]*, we have*

$$p_1 p_2 \in C^\tau_* S^{m_1+m_2}_{1,\delta},$$

where $\delta = \max\{\delta_1, \delta_2\}$. We can easily extend the inequality (2.6) *to the case of n functions, i.e. if $u_i \in C^\tau_*$ for $1 < i \le n$, then*

$$\|u_1 u_2 ... u_n\|_{C^\tau_*} \le \sum_{j=1}^{n} \|u_j\|_{C^\tau_*}\Big(\Pi^n_{\substack{i=1\\ i\neq j}} \|u_i\|_{L^\infty}\Big).$$

Lemma 2.1.9. *a) Let $\tau \in \mathbb{R}_{>0}$. Then $\langle D\rangle := (1-\Delta)^{\frac{1}{2}} : C^\tau_* \to C^{\tau-1}_*$ (Δ=Laplace operator) is an isomorphism.*

b) Let $\tau \in \mathbb{R}_{>0}$. Then there exist constants $C_1, C_2 > 0$ such that $C_1\|f\|_{C_*^\tau} \leq \|\langle D\rangle f\|_{C_*^{\tau-1}} \leq C_2\|f\|_{C_*^\tau}$ for all $f \in C_*^\tau$.

c) Let $\tau \in \mathbb{R}_{>0}$. Then there exists a constant $C > 0$ such that

$$\|\langle D\rangle f\|_{C_*^{\tau-1}} \leq C\Big(\|f\|_{C_*^{\tau-1}} + \sum_{j=1}^{n} \|\partial_{x_j} f\|_{C_*^{\tau-1}}\Big).$$

for all $f \in C_*^\tau$.

Proof. a) Cf. [24, Chapter 13, Section 8].

b) Since $\langle D\rangle$ is an isomorphism.

c)

$$\begin{aligned}\langle D\rangle &= (1-\Delta)^{\frac{1}{2}} = \frac{1-\Delta}{\sqrt{1-\Delta}} \\ &= \frac{1}{\sqrt{1-\Delta}} - \frac{\partial_{x_1}}{\sqrt{1-\Delta}}\partial_{x_1} - \dots - \frac{\partial_{x_n}}{\sqrt{1-\Delta}}\partial_{x_n}.\end{aligned}$$

$\frac{\partial_{x_j}}{\sqrt{1-\Delta}}$ for $j = 1, .., n$ has zero order. Then we have

$$\begin{aligned}\|f\|_{C_*^\tau} \overbrace{\cong}^{b)} \|\langle D\rangle f\|_{C_*^{\tau-1}} &= \|\langle D\rangle^{-1} f - \frac{\partial_{x_1}}{\sqrt{1-\Delta}}\partial_{x_1} f - \dots - \frac{\partial_{x_n}}{\sqrt{1-\Delta}}\partial_{x_n} f\|_{C_*^{\tau-1}} \\ &\leq C\Big(\|f\|_{C_*^{\tau-1}} + \|\partial_{x_1} f\|_{C_*^{\tau-1}} + \dots + \|\partial_{x_n} f\|_{C_*^{\tau-1}}\Big) \\ &= C\Big(\|f\|_{C_*^{\tau-1}} + \sum_{j=1}^{n} \|\partial_{x_j} f\|_{C_*^{\tau-1}}\Big).\end{aligned}$$

□

Lemma 2.1.10. *If $f \in C^\tau(\mathbb{R}^n)(C_*^\tau(\mathbb{R}^n))$ and there exists $c > 0$ such that $f \geq c > 0$, then $\sqrt{f} \in C^\tau(\mathbb{R}^n)(C_*^\tau(\mathbb{R}^n))$.*

Proof. For $\tau \in \mathbb{Z}_{\geq 0}$ this is obvious.
Let $0 < \tau < 1$ and $f \in C^\tau$. Then there exists $C > 0$ such that

$$|f(x) - f(y)| \leq C|x-y|^\tau.$$

$$|\sqrt{f(x)} - \sqrt{f(y)}| = \frac{|f(x) - f(y)|}{\sqrt{f(x)} + \sqrt{f(y)}} \leq \frac{C|x-y|^\tau}{2\sqrt{c}}.$$

Therefore

$$\frac{|\sqrt{f(x)} - \sqrt{f(y)}|}{|x-y|^\tau} \leq \frac{C}{2\sqrt{c}}.$$

Next we show that $\left(\sqrt{f}\right)^{-1} \in C^\tau$ for $0 < \tau < 1$.

$$\Big|\left(\sqrt{f(x)}\right)^{-1} - \left(\sqrt{f(y)}\right)^{-1}\Big| = \Big|\frac{1}{\sqrt{f(x)}}\left(\sqrt{f(x)} - \sqrt{f(y)}\right)\frac{1}{\sqrt{f(y)}}\Big| \leq \frac{C|x-y|^\tau}{c^2}.$$

Therefore

$$\frac{\left|\left(\sqrt{f(x)}\right)^{-1} - \left(\sqrt{f(y)}\right)^{-1}\right|}{|x-y|^\tau} \le \frac{C}{c^2}.$$

Now let $\tau = 1 + \sigma$, and assume that it has been shown that $\sqrt{f} \in C^\sigma$ and $\left(\sqrt{f}\right)^{-1} \in C^\sigma$ whenever $f \in C^\tau$ and $f \ge c > 0$. If we set $f(x) = \frac{1}{\sqrt{x}} \in C^\tau$, $g = f \in C^\tau$ in Lemma 2.1.5, then from (2.4) we have

$$\left\|\left(\sqrt{f}\right)^{-1}\right\|_{C^\tau} \le C\left(c'\|f\|_{C^1}^\tau + c'\|f\|_{C^\tau} + c'\right) \le C'.$$

Therefore $\left(\sqrt{f}\right)^{-1} \in C^\tau$. From (2.5)

$$\begin{aligned}
\left\|\sqrt{f}\right\|_{C^\tau} &\le \left\|\sqrt{f}\right\|_{C^\sigma} + \sum_{j=1}^n \left\|\partial_{x_j}\sqrt{f}\right\|_{C^\sigma} \\
&\le C' + C'\left(\left\|\left(\sqrt{f}\right)^{-1}\right\|_{C^\sigma}\|\partial_{x_j} f\|_{L^\infty} + \left\|\left(\sqrt{f}\right)^{-1}\right\|_{L^\infty}\|\partial_{x_j} f\|_{C^\sigma}\right) \\
&= C' + C'\left(\left\|\left(\sqrt{f}\right)^{-1}\right\|_{C^\sigma}\|\partial_{x_j} f\|_{L^\infty} + \left\|\left(\sqrt{f}\right)^{-1}\right\|_{L^\infty}\|f\|_{C^{\sigma+1}}\right) \\
&\le C''
\end{aligned}$$

□

Remark 2.1.11. *The symbol smoothing technique has been introduced by Taylor for scalar symbols. It extends to the vector-valued setting since it is based on elementary estimates: If* $p \in C^\tau_* S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; X)$, *for* $\delta \in [0,1)$, *then for every* $\gamma \in (\delta, 1)$ *there is a decomposition*

$$p(x,\xi) = p^\sharp(x,\xi) + p^b(x,\xi) \tag{2.7}$$

with

$$p^\sharp \in S^m_{1,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; X)\,,\ p^b \in C^\tau_* S^{m-(\gamma-\delta)\tau}_{1,\gamma}(\mathbb{R}^n \times \mathbb{R}^n; X).$$

The symbol decomposition (2.7) *is constructed as follows. Use the Littlewood-Paley partition of unity* φ_h *, and set*

$$p^\sharp(x,\xi) = \sum_{h=0}^\infty J_{\epsilon_h} p(x,\xi)\varphi_h(\xi),$$

where J_ϵ *is a smoothing operator on functions of x, namely*

$$J_\epsilon f(x) = \psi(\epsilon D) f(x),$$

with $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi(\xi) = 1$ *for* $|\xi| < 1$ *(e.g.,* $\psi = \varphi_0$*), and we take*

$$\epsilon_h = 2^{-h(\gamma-\delta)}. \tag{2.8}$$

We define $p^b(x,\xi)$ *to be* $p(x,\xi) - p^\sharp(x,\xi)$, *yielding* (2.7). *Cf.* [22, p. 51, 52]. *To analyze this terms, we use the following lemma.*

Lemma 2.1.12. *Let* $f \in C^\tau$, $\tau > 0$. *Then we have* *a)*

$$\|D_x^\beta J_\epsilon f\|_{L^\infty} \leq C\|f\|_{C^{|\beta|}} \leq C\|f\|_{C^\tau} \qquad \textit{for } |\beta| \leq \tau \tag{2.9}$$

$$\|D_x^\beta J_\epsilon f\|_{L^\infty} \leq C\epsilon^{-(|\beta|-\tau)}\|f\|_{C_*^\tau} \qquad \textit{for } |\beta| > \tau. \tag{2.10}$$

b)

$$\|D_x^\beta J_\epsilon f\|_{C_*^\tau} \leq C_\beta\ \epsilon^{-|\beta|}\|f\|_{C_*^\tau}. \tag{2.11}$$

c)

$$\|f - J_\epsilon f\|_{C_*^{\tau-t}} \leq C\epsilon^t\|f\|_{C_*^\tau} \qquad \textit{for } t \geq 0. \tag{2.12}$$

d)

$$\|f - J_\epsilon f\|_{L^\infty} \leq C_\tau\epsilon^\tau\|f\|_{C_*^\tau}. \tag{2.13}$$

Proof. a) For $|\beta| \leq \tau$:

$$\begin{aligned} \|D_x^\beta J_\epsilon f\|_{L^\infty} = \|J_\epsilon D_x^\beta f\|_{L^\infty} &= \|\epsilon^{-n}\int (D_x^\beta f)(y-x)\hat{\psi}(\frac{y}{\epsilon})dy\|_{L^\infty} \\ &\leq \|D_x^\beta f\|_\infty \int |\epsilon^{-n}\hat{\psi}(\frac{y}{\epsilon})|\,dy \leq C\|f\|_{C^{|\beta|}} \leq C\|f\|_{C^\tau} \end{aligned}$$

For $|\beta| > \tau$: Cf. [22, Chapter 1, Section 1.3, Lemma 1.3.C].

b) Cf. [22, Chapter 13, Section 9, Lemma 9.8].

c) Cf. [22, Chapter 13, Section 9, Lemma 9.8].

d) Cf. [22, Chapter 13, Section 9, Lemma 9.8].

□

Lemma 2.1.13. *Let $p \in C_*^\tau S_{1,\delta}^m$, $\tau > 0$. Then in the decomposition* (2.7)*,*

$$p^\sharp(x,\xi) \in S_{1,\gamma}^m \tag{2.14}$$

such that

$$\partial_x^\beta p^\sharp \in S_{1,\gamma}^{m+\delta|\beta|}(\mathbb{R}^n \times \mathbb{R}^n; X)\ for\ |\beta| \leq \tau, \tag{2.15}$$

$$\partial_x^\beta p^\sharp \in S_{1,\gamma}^{m+(\gamma-\delta)(|\beta|-\tau)+\delta\tau}(\mathbb{R}^n \times \mathbb{R}^n; X)\ for\ |\beta| > \tau, \tag{2.16}$$

and

$$p^b(x,\xi) \in C_*^\tau S_{1,\gamma}^{m-(\gamma-\delta)\tau}. \tag{2.17}$$

Proof. For $|\beta| \leq \tau$: We first recall that φ_h is supported in $\left\{2^{h-1} \leq |\xi| \leq 2^{h+1}\right\}$. Hence

$$D_x^\beta p^\sharp(x,\xi) = \sum_{h=0}^\infty \binom{\alpha}{\alpha'} J_{\epsilon_h}\Big(\partial_x^\beta p(x,\xi)\Big)\varphi_h = \Big(\overbrace{D_x^\beta p(x,\xi)}^{C_*^{\tau-|\beta|}S_{1,\delta}^{m+\delta|\beta|}}\Big)^\sharp \in S_{1,\gamma}^{m+\delta|\beta|}$$

For $|\beta| > \tau$:

$$
\begin{aligned}
\|D_\xi^\alpha \partial_x^\beta p^\sharp(.,\xi)\|_{L^\infty} &= \|\sum_{h=0}^{\infty}\sum_{\alpha'\leq\alpha}\binom{\alpha}{\alpha'} J_{\epsilon_h} D_\xi^{\alpha'}\partial_x^\beta p(.,\xi) D_\xi^{\alpha-\alpha'}\varphi_h\|_{L^\infty} \\
&\overbrace{\leq}^{(2.10)} C\sup_h \epsilon_h^{-(|\beta|-\tau)}\|D_\xi^{\alpha'} p(.,\xi)\|_{C_*^\tau}\|D_\xi^{\alpha-\alpha'}\varphi_h\|_{L^\infty} \\
&\overbrace{\leq}^{(2.8)} C\sup_h 2^{h(\gamma-\delta)(|\beta|-\tau)}\langle\xi\rangle^{m-|\alpha|+\delta\tau} \text{ on } \left\{|\xi| \sim 2^h\right\} \\
&\leq C\langle\xi\rangle^{m-|\alpha|+\delta\tau+(\gamma-\delta)(|\beta|-\tau)}.
\end{aligned}
$$

Therefore $p^\sharp(x,\xi) \in S^m_{1,\gamma}$. Details on $p^b(x,\xi) \in C_*^\tau S_{1,\gamma}^{m-(\gamma-\delta)\tau}$ can be found in Cf. [1, Definition A.1] and Cf. [23, Chapter 1, Section 3, Proposition 3.2].

□

Theorem 2.1.14. *a) If $p \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, then $p(x, D_x)$ extends to a bounded linear operator*

$$p(x, D_x) : H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

for all $s \in \mathbb{R}$.

b) Let $\tau > 0$, $0 \leq \delta < 1$. If $p \in C_*^\tau S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, then $p(x, D_x)$ extends to a bounded linear operator

$$p(x, D_x) : H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

for every $-\tau(1-\delta) < s < \tau$.

Proof. a) Cf. [24, Chapter 7, Proposition 5.5].

b) Cf. [22, Chapter 13, Proposition 9.10].

□

Corollary 2.1.15. *For every $\tau > 0$ and $-\tau < s < \tau$ there exists a positive constant C such that*

$$\|uv\|_{H^s} \leq C\|u\|_{H^s}\|v\|_{C^\tau}, \tag{2.18}$$

is true for any $u \in H^s(\mathbb{R}^n)$ and $v \in C^\tau(\mathbb{R}^n)$.

Proof. Cf. [6, Corollary 4.5]. □

Theorem 2.1.16. *Let $p \in S^{m_1}_{1,\delta_1}$ or $p \in C_*^\tau S^{m_1}_{1,\delta_1}$ and $q \in S^{m_2}_{1,\delta_2}$, $m_1, m_2 \in \mathbb{R}$, $\tau > 0$ and $0 \leq \delta_1, \delta_2 < 1$. We denote by $p\sharp q$ the symbol of $p(x, D_x)q(x, D_x)$. It has the asymptotic expansion*

$$p\sharp q \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} D_\xi^\alpha p \partial_x^\alpha q.$$

Proof. Cf. [9, Chapter 7, Section 7.3, Theorem 7.13]. □

Definition 2.1.17. *Let $p \in C_*^{\tau_1} S_{1,\delta_1}^{m_1}$ and $q \in C_*^{\tau_2} S_{1,\delta_2}^{m_2}$ such that $m_1, m_2 \in \mathbb{R}$, $0 \leq \delta_1, \delta_2 < 1$ and $\tau_1, \tau_2 > 0$. We define $p\sharp_{[\theta]}q$ for $\theta \in (0, \tau_2)$ as follows:*

$$p\sharp_{[\theta]}q := \sum_{|\alpha| \leq [\theta]} \frac{1}{\alpha!} D_\xi^\alpha p \partial_x^\alpha q.$$

Cf. [1, p. 1473].

Theorem 2.1.18. *Let $\tau_2, \tau_1 > 0$, $m_1, m_2 \in \mathbb{R}$, $0 \leq \delta_1, \delta_2 < 1$, $\theta \in (0, \tau_2)$, $\frac{(1-\delta_2)\theta}{\tau_2} + \delta_2 > \delta_1$ and $\tau := \min\{\tau_1, \tau_2 - [\theta]\}$. If $p_1 \in C_*^{\tau_1} S_{1,\delta_1}^{m_1}(\mathbb{R}^n \times \mathbb{R}^n)$ and $p_2 \in C_*^{\tau_2} S_{1,\delta_2}^{m_2}(\mathbb{R}^n \times \mathbb{R}^n)$, then for every $s \in \mathbb{R}$ such that $-\tau(1-\delta_1) < s < \tau$, $-(1-\delta_2)(\tau_2 - \theta) < s + m_1 < \tau_2$, and $-(1-\delta_2)(\tau_2 - [\theta]) < s$*

$$\mathcal{R}(x, D_x) := p_1(x, D_x)p_2(x, D_x) - p_1\sharp_{[\theta]}p_2(x, D_x) : H^{s+m_1+m_2-(1-\delta_2)\theta}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

is a bounded operator.

Proof. Let $\gamma := \frac{(1-\delta_2)\theta}{\tau_2} + \delta_2$. By Remark 2.1.11 and Lemma 2.1.13 we have the following decompositions for p_1 and p_2:

$$p_1(x, \xi) = p_1^\sharp(x, \xi) + p_1^b(x, \xi)$$

with $p_1^b \in C_*^{\tau_1} S_{1,\gamma}^{m_1-(\gamma-\delta_1)\tau_1}$, $p_1^\sharp \in S_{1,\gamma}^{m_1}$ and

$$\begin{aligned} \partial_x^\alpha p_1^\sharp(x, \xi) &\in S_{1,\gamma}^{m_1+\delta_1\tau_1}(\mathbb{R}^n \times \mathbb{R}^n) \text{ if } |\alpha| \leq \tau_1 \\ \partial_x^\alpha p_1^\sharp(x, \xi) &\in S_{1,\gamma}^{m_1+(\gamma-\delta_1)(|\alpha|-\tau_1)+\delta_1\tau_1}(\mathbb{R}^n \times \mathbb{R}^n) \text{ if } |\alpha| > \tau_1, \end{aligned}$$

$$p_2(x, \xi) = p_2^\sharp(x, \xi) + p_2^b(x, \xi)$$

with $p_2^b \in C_*^{\tau_2} S_{1,\gamma}^{m_2-(\gamma-\delta_2)\tau_2} = C_*^{\tau_2} S_{1,\gamma}^{m_2-(1-\delta_2)\theta}$, $p_2^\sharp \in S_{1,\gamma}^{m_2}$ and

$$\begin{aligned} \partial_x^\alpha p_2^\sharp(x, \xi) &\in S_{1,\gamma}^{m_2+\delta_2\tau_2}(\mathbb{R}^n \times \mathbb{R}^n) \text{ if } |\alpha| \leq \tau_2 \\ \partial_x^\alpha p_2^\sharp(x, \xi) &\in S_{1,\gamma}^{m_2+(\gamma-\delta_2)(|\alpha|-\tau_2)+\delta_2\tau_2} = S_{1,\gamma}^{m_2-(1-\delta_2)\theta-\delta_2(|\alpha|-\tau_2)+\gamma|\alpha|}(\mathbb{R}^n \times \mathbb{R}^n) \text{ if } |\alpha| > \tau_2. \end{aligned}$$

Then

$$p_1(x, D_x)p_2(x, D_x) = \overbrace{p_1(x, D_x)p_2^\sharp(x, D_x)}^{\text{(II)}} + \overbrace{p_1(x, D_x)p_2^b(x, D_x)}^{\text{(I)}}.$$

We will estimate (I) and (II) separately.

(I) As $p_1 \in C_*^{\tau_1} S_{1,\delta_1}^{m_1}$ and $p_2^b \in C_*^{\tau_2} S_{1,\gamma}^{m_2-(1-\delta_2)\theta}$ we conclude from Theorem 2.1.14 that

$$p_1(x, D_x) : H^{s+m_1}(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \text{ for } -\tau_1(1-\delta_1) < s < \tau_1 \textit{ and}$$

$$p_2^b(x, D_x) : H^{s+m_2+m_1-(1-\delta_2)\theta}(\mathbb{R}^n) \to H^{s+m_1}(\mathbb{R}^n) \text{ for } -(1-\delta_2)(\tau_2 - \theta) < s + m_1 < \tau_2$$

are bounded operators. Therefore

$$p_1(x,D_x)p_2^b(x,D_x) : H^{s+m_2+m_1-(1-\delta_2)\theta}(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \tag{2.19}$$

is a bounded operator since $-(1-\delta_2)(\tau_2-\theta) < s+m_1 < \tau_2$ and $-\tau_1(1-\delta_1) < s < \tau_1$.

(II) $p_1(x,D_x)p_2^\sharp(x,D_x) = p_1^\sharp(x,D_x)p_2^\sharp(x,D_x) + p_1^b(x,D_x)p_2^\sharp(x,D_x)$. By definition p_1^b, $p_1^\sharp$ are smooth in ξ and $p_2^\sharp$ is smooth in x. By Theorem 2.1.16 and Definition 2.1.17 $p_1^b\sharp p_2^\sharp$ and $p_1^\sharp\sharp p_2^\sharp$ are the symbols of $p_1^b(x,D_x)p_2^\sharp(x,D_x)$ and $p_1^\sharp(x,D_x)p_2^\sharp(x,D_x)$ respectively, with

$$p_1^b\sharp p_2^\sharp \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} D_\xi^\alpha p_1^b(x,\xi)\partial_x^\alpha p_2^\sharp(x,\xi),$$

$$p_1^\sharp\sharp p_2^\sharp \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} D_\xi^\alpha p_1^\sharp(x,\xi)\partial_x^\alpha p_2^\sharp(x,\xi).$$

For $|\alpha| > \theta$, the fact that

$$\begin{aligned} D_\xi^\alpha p_1^b &\in C_*^{\tau_1} S_{1,\gamma}^{m_1-(\gamma-\delta_1)\tau_1-|\alpha|}, \\ \partial_x^\alpha p_2^\sharp &\in \begin{cases} S_{1,\gamma}^{m_2+\delta_2|\alpha|} & |\alpha| \le \tau_2 \\ S_{1,\gamma}^{m_2-(1-\delta_2)\theta-\delta_2(|\alpha|-\tau_2)+\gamma|\alpha|} & |\alpha| > \tau_2 \end{cases}, \\ D_\xi^\alpha p_1^\sharp &\in S_{1,\gamma}^{m_1-|\alpha|}, \end{aligned}$$

implies that

$$D_\xi^\alpha p_1^b \partial_x^\alpha p_2^\sharp \in \begin{cases} C_*^{\tau_1} S_{1,\gamma}^{m_1+m_2-(1-\delta_2)|\alpha|-(\gamma-\delta_1)\tau_1} & \theta < |\alpha| \le \tau_2 \\ \subset C_*^{\tau_1} S_{1,\gamma}^{m_1+m_2-(1-\delta_2)\theta-(\gamma-\delta_1)\tau_1} & \\ & \\ C_*^{\tau_1} S_{1,\gamma}^{m_1+m_2-(1-\delta_2)\theta-(\gamma-\delta_1)\tau_1-\delta_2(|\alpha|-\tau_2)-|\alpha|(1-\gamma)} & |\alpha| > \tau_2, |\alpha| > \theta \\ \subset C_*^{\tau_1} S_{1,\gamma}^{m_1+m_2-(1-\delta_2)\theta-(\gamma-\delta_1)\tau_1-|\alpha|(1-\gamma)} & \end{cases} \tag{2.20}$$

$$D_\xi^\alpha p_1^\sharp \partial_x^\alpha p_2^\sharp \in \begin{cases} S_{1,\gamma}^{m_1+m_2-(1-\delta_2)|\alpha|} & \theta < |\alpha| \le \tau_2 \\ \subset S_{1,\gamma}^{m_1+m_2-(1-\delta_2)\theta} & \\ & \\ S_{1,\gamma}^{m_1+m_2-(1-\delta_2)\theta-\delta_2(|\alpha|-\tau_2)-|\alpha|(1-\gamma)} & |\alpha| > \tau_2, |\alpha| > \theta \\ \subset S_{1,\gamma}^{m_1+m_2-(1-\delta_2)\theta-|\alpha|(1-\gamma)} & \end{cases} \tag{2.21}$$

Therefore

$$p_1\sharp p_2^\sharp = \sum_{|\alpha|\le[\theta]} \frac{1}{\alpha!} \overbrace{D_\xi^\alpha p_1(x,\xi)\partial_x^\alpha p_2^\sharp(x,\xi)}^{\text{(III)}} + \mathcal{R}^\sharp(x,\xi) + \mathcal{R}^b(x,\xi). \tag{2.22}$$

Here we have used the estimates (2.20) and (2.21) for $|\alpha| > \theta$ and the fact that $-(1-\gamma)|\alpha| \le 0$. By Theorem 2.1.16 and Definition 2.1.17

$$\mathcal{R}^\sharp(x,\xi) = p_1^\sharp \sharp p_2^\sharp - p_1^\sharp \sharp_{[\theta]} p_2^\sharp \sim \sum_{|\alpha|>[\theta]} \frac{1}{\alpha!} D_\xi^\alpha p_1^\sharp \partial_x^\alpha p_2^\sharp \in S_{1,\gamma}^{m_1+m_2-(1-\delta_2)\theta}$$

$$\mathcal{R}^b(x,\xi) = p_1^b \sharp p_2^\sharp - p_1^b \sharp_{[\theta]} p_2^\sharp \sim \sum_{|\alpha|>[\theta]} \frac{1}{\alpha!} D_\xi^\alpha p_1^b \partial_x^\alpha p_2^\sharp \in C_*^{\tau_1} S_{1,\gamma}^{m_1+m_2-(1-\delta_2)\theta-(\gamma-\delta_1)\tau_1}$$

By Theorem 2.1.14

$$\mathcal{R}^\sharp(x,D_x) : H^{s+m_1+m_2-(1-\delta_2)\theta}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

is bounded for all $s \in \mathbb{R}$ and

$$\mathcal{R}^b(x,D_x) : H^{s+m_1+m_2-(1-\delta_2)\theta}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

for all $-\tau_1(1-\delta_1) < s < \tau_1$ is a bounded operator. Hence

$$\mathcal{R}^\sharp(x,D_x) + \mathcal{R}^b(x,D_x) : H^{s+m_1+m_2-(1-\delta_2)\theta}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

is bounded if $-\tau_1(1-\delta_1) < s < \tau_1$.

(III) Now we want to replace $p_2^\sharp$ by p_2 in (III). Clearly

$$\begin{aligned} D_\xi^\alpha p_1(x,\xi)\partial_x^\alpha p_2^\sharp(x,\xi) &= D_\xi^\alpha p_1(x,\xi)\partial_x^\alpha p_2(x,\xi) \\ &\quad - \overbrace{D_\xi^\alpha p_1^\sharp(x,\xi)\partial_x^\alpha p_2^b(x,\xi)}^{(IV)} - \overbrace{D_\xi^\alpha p_1^b(x,\xi)\partial_x^\alpha p_2^b(x,\xi)}^{(V)} \end{aligned} \tag{2.23}$$

We will estimate (2.23) and insert it in (2.22).

(IV) $D_\xi^\alpha p_1^\sharp \partial_x^\alpha p_2^b \in C_*^{\tau_2-|\alpha|} S_{1,\gamma}^{m_1+m_2-(1-\delta_2)\theta-|\alpha|(1-\gamma)}$, hence, since $|\alpha|(1-\gamma) \ge 0$

$Op(D_\xi^\alpha p_1^\sharp \partial_x^\alpha p_2^b) : H^{s+m_1+m_2-(1-\delta_2)\theta-|\alpha|(1-\gamma)}(\mathbb{R}^n) \subset H^{s+m_1+m_2-(1-\delta_2)\theta}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$,

is a bounded operator for $-(\tau_2-[\theta])(1-\delta_2) < s < \tau_2-[\theta]$.

(V) Since $|\alpha|(1-\gamma) \ge 0$

$$D_\xi^\alpha p_1^b \partial_x^\alpha p_2^b \in C_*^\tau S_{1,\gamma}^{m_1+m_2-(1-\delta_2)\theta-|\alpha|(1-\gamma)-(\gamma-\delta_1)\tau_1} \subseteq C_*^\tau S_{1,\gamma}^{m_1+m_2-(1-\delta_2)\theta},$$

and

$$Op(D_\xi^\alpha p_1^b \partial_x^\alpha p_2^b) : H^{m_1+m_2-(1-\delta_2)\theta}(\mathbb{R}^n) \to H^s(\mathbb{R}^n),$$

for all $-\tau(1-\delta_1) < s < \tau$ is a bounded operator. Therefore, by combining all terms, we conclude that

$$p_1(x,D_x)p_2(x,D_x) - p_1\sharp_{[\theta]}p_2(x,D_x) : H^{s+m_1+m_2-(1-\delta_2)\theta}(\mathbb{R}^n) \to H^s(\mathbb{R}^n),$$

is bounded for $-\tau(1-\delta_1) < s < \tau$, $-(1-\delta_2)(\tau_2-\theta) < s+m_1 < \tau_2$ and $-(1-\delta_2)(\tau_2-[\theta]) < s$. □

Remark 2.1.19. *In* [15, Theorem 3], *Marschall stated a similar theorem without proof. This provides a proof for this result.*

We denote by $S(\overline{\mathbb{R}}_+)$ the space of all restrictions of functions in $S(\mathbb{R})$ to $\overline{\mathbb{R}}_+$.

Definition 2.1.20. *The space $C^\tau_* S^m_{1,\delta}(\mathbb{R}^N \times \mathbb{R}^{n-1}, S(\overline{\mathbb{R}}_+))$, $m \in \mathbb{R}$, n, $N \in \mathbb{N}$, consists of all functions $\tilde{f}(x, \xi', y_n)$, which are smooth in $(\xi', y_n) \in \mathbb{R}^{n-1} \times \overline{\mathbb{R}}_+$, are in $C^\tau_*(\mathbb{R}^{\mathbb{N}})$ with respect to x, and satisfy*

$$\sup_{x' \in \mathbb{R}^N} \|y_n^l \partial_{y_n}^{l'} D^\alpha_{\xi'} \tilde{f}(.,\xi',.)\|_{L^2_{y_n}(\mathbb{R}_+)} \leq C_{\alpha,l,l'} \langle \xi' \rangle^{m+\frac{1}{2}-l+l'-|\alpha|} \tag{2.24}$$

$$\|y_n^l \partial_{y_n}^{l'} D^\alpha_{\xi'} \tilde{f}(.,\xi',.)\|_{C^\tau_*(\mathbb{R}^N; L^2_{y_n}(\mathbb{R}_+))} \leq C_{\alpha,l,l'} \langle \xi' \rangle^{m+\frac{1}{2}-l+l'-|\alpha|+\delta\tau} \tag{2.25}$$

for all $\alpha \in \mathbb{N}_0^{n-1}$, l, $l' \in \mathbb{N}_0$. Cf. [2, Definition A.3].

Definition 2.1.21. *Let $k = k(x, \xi', y_n) \in C^\tau_* S^{m-1}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n-1}, S(\overline{\mathbb{R}}_+))$, $m \in \mathbb{R}$. Then we define the Poisson operator of order d by*

$$k(x, D_x)a = \mathcal{F}^{-1}_{\xi' \mapsto x'}[k(x, \xi', y_n) \mathcal{F}_{x' \mapsto \xi'} a(\xi')],\ a \in S(\mathbb{R}^{n-1}) .$$

k is called a Poisson symbol-kernel of order m. Cf. [2, Definition A.4]

Theorem 2.1.22. *Let $k(x, \xi', y_n) \in C^\tau_* S^{m-1}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n-1}, S(\overline{\mathbb{R}}_+))$, $m \in \mathbb{R}$. Then $k(x, D_x)$ extends to a bounded operator*

$$k(x', D_x) : H^{s+m-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^n_+)$$

for every $-\tau(1-\delta) < s < \tau$.

Proof. Cf. [2, Theorem A.5]. □

2.2 Transmission Condition for a Non-Smooth Symbol

In this section we consider a transmission condition for non-smooth pseudodifferential operators. In order to understand the transmission condition we need a characterization of the spaces $\mathcal{F}[e^\pm S(\overline{\mathbb{R}}_\pm)]$, where $e^\pm$ are the extension by zero operator. In order to characterize these spaces we have to introduce the following Fréchet spaces H^+, H^-_0 and H_d.

We recall that $S(\overline{\mathbb{R}}_\pm)$ is the space of all restrictions of functions in the Schwartz space $S(\mathbb{R})$ to $\overline{\mathbb{R}}_\pm$ and that both are nuclear Fréchet spaces. We denote by H^+ and H^-_0 the following spaces:

$$\begin{aligned} H^+ &= \left\{ \mathcal{F}(e^+ u) : u \in S(\overline{\mathbb{R}}_+) \right\}, \\ H^-_0 &= \left\{ \mathcal{F}(e^- u) : u \in S(\overline{\mathbb{R}}_-) \right\}, \end{aligned}$$

where $e^\pm$ are the operators of extension-by-zero operators from $\overline{\mathbb{R}}_\pm$ to $\mathbb{R}$:

$$[e^- f](t) = \begin{cases} f(t) & t \leq 0 \\ 0 & t > 0 \end{cases},$$

$$[e^+ f](t) = \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases} .$$

We denote by H'_d the space of all polynomials of degree $\leq d-1$ and let

$$H_d = H_0 \oplus H'_d = H^+ \oplus H_0^- \oplus H'_d = \mathcal{F}[e^+ S(\overline{\mathbb{R}}_+)] \oplus \mathcal{F}[e^- S(\overline{\mathbb{R}}_-)] \oplus H'_d.$$

H_d ($d \in \mathbb{N}_0$) consists of smooth functions $h : \mathbb{R} \to \mathbb{C}$ which admit an asymptotic expansion $h(t) \sim s_d t^d + s_{d-1} t^{d-1} + ...$ such that

$$\left|\partial_t^l [t^k h(t) - \sum_{j=d-N}^{d} s_j t^{j+k}]\right| \leq C_{k,l,N}(1 + |t|)^{d-N-1+k-l} \tag{2.26}$$

for all k, l and $N \in \mathbb{N}$, as $|t| \to \infty$. The operators $h_+ = \mathcal{F} e^+ r^+ \mathcal{F}^{-1}$ and $h_- = \mathcal{F} e^- r^- \mathcal{F}^{-1}$ are continuous projections on H^+ and H_0^-, respectively. Moreover

$$h_0 : H_d = H_0 \oplus H'_d \to H_0$$

is the projection onto the first summand. Cf. [16, Definition 2.1].

Definition 2.2.1. *Let $m \in \mathbb{Z}$. A symbol $p \in C^\tau_* S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is said to satisfy the transmission condition, if there are $s_{k,\alpha}(x, \xi')$, smooth in ξ' and in C^τ_* w.r.t. x, such that for any $\alpha \in \mathbb{N}_0^n$ and $l \in \mathbb{N}_0$*

$$\left\|\xi_n^l D_\xi^\alpha p(\cdot, \xi) - \sum_{k=-l}^{m-|\alpha|} s_{k,\alpha}(\cdot, \xi') \xi_n^{k+l}\right\|_{C^\tau_*(\mathbb{R}^n)} \leq C_{k,l,\alpha} \langle \xi' \rangle^{m+1+l-|\alpha|} |\xi_n|^{-1}, \tag{2.27}$$

when $|\xi_n| \geq \langle \xi' \rangle$. The functions $s_{k,\alpha}$ have to be zero after a finite number of differentiations in ξ'. Hence, they are polynomial in ξ' of degree $m - k - |\alpha|$ with coefficients in $C^\tau_*(\mathbb{R}^n)$. Cf. [1, Definition 5.2].

Lemma 2.2.2. *Let $m \in \mathbb{Z}$ and $p \in C^\tau_* S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy the transmission condition. Then $r^+ p(x, D_x)(\delta \otimes a) = k(x, D_x)a$ for all $a \in S(\mathbb{R}^{n-1})$, where $k \in C^\tau_* S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, S(\overline{\mathbb{R}}_+))$ is a Poisson symbol-kernel of order $m + 1$. Here δ is the delta distribution with respect to x_n.*

Proof. Cf. [1, Lemma 5.4]. □

Remark 2.2.3. *An important tool in the calculus are "order-reducing operators". There are two types, one acting over the domain and one acting over the boundary:*

$$\Lambda^m_{-,\mathbb{R}^n_+} = \mathrm{Op}(r^m_-(\xi))_+ : H^s(\mathbb{R}^n_+) \to H^{s-m}(\mathbb{R}^n_+)$$

$$\Lambda^t_0 = \mathrm{Op}'(\langle \xi' \rangle) : H^s(\mathbb{R}^{n-1}) \to H^{s-t}(\mathbb{R}^{n-1}),\ \forall s \in \mathbb{R}\ ,$$

$m \in \mathbb{Z}$ and $t \in \mathbb{R}$. Here $\Lambda^m_{-,\mathbb{R}^n_+}$ and Λ^t_0 have inverses $\Lambda^{-m}_{-,\mathbb{R}^n_+}$ and Λ^{-t}_0. Cf. [8, Chapter 2, Section 2.5].

Definition 2.2.4. *Let $m \in \mathbb{R}$ and $r \in \mathbb{N}_0$.*

1. *If* $t \in C^\tau_* S^m_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, S(\overline{\mathbb{R}}_+))$, $s_j \in C^\tau_* S^{m-j}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, $j = 0, ..., r-1$, *then the associated trace operator of order d and class r is defined as*

$$\begin{aligned} t(x', D_x)f &= \sum_{j=0}^{r-1} s_j(x', D_{x'})\gamma_j f + t_0(x', D_x)f, \\ t_0(x', D_x)f &= \mathcal{F}^{-1}_{\xi \to x'}\Big[\int_0^\infty t_0(x', \xi', y_n)\hat{f}(\xi', y_n)dy_n\Big], \end{aligned}$$

where $t_0 \in C^\tau_* S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; S(\overline{\mathbb{R}}_+))$ *and* $\gamma_j f = \partial^j_n f|_{x_n=0}$.

2. *If* $g \in C^\tau_* S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, S(\overline{\mathbb{R}}^2_{++}))$, $k_j \in C^\tau_* S^{m-j-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, S(\overline{R}_+))$ *for* $j = 0, ..., r-1$, *then the associated singular Green operator of order d and class r is defined as*

$$\begin{aligned} g(x, D_x)f &= \sum_{j=0}^{r-1} k_j(x, D_{x'})\gamma_j f + g_0(x, D_x)f, \\ g_0(x, D_x)f &= \mathcal{F}^{-1}_{\xi' \to x'}\Big[\int_0^\infty g_0(x, \xi', y_n)\hat{f}(\xi', y_n)dy_n\Big]. \end{aligned}$$

where $g_0 \in C^\tau_* S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, S(\overline{\mathbb{R}}^2_{++}))$ *and* $\gamma_j f$ *is as above.*

The operators $t(x, D_x)$ *and* $g(x, D_x)$ *are said to be class of* $-r$ *for* $r \in \mathbb{N}$ *if* $t(x, \xi), g(x, \xi) \in H_{r-1}$. *Cf.* [1, Definition 4.4 and Remark 5.1].

Example 2.2.5. *Let* γ_j *be a trace operator. For* $m \in \mathbb{Z}$ *by Grubb,* [8, Theorem 2.8.1] *we have* $\gamma_j \mathrm{Op}(r^m_-(\xi))_+$ *of order* $j + m + \frac{1}{2}$ *and class* $j + m + 1$.

2.3 Differential Operator with C^τ_* Coefficients

Let us now introduce the notions of ellipticity and uniform strong ellipticity. Since we will study in Section 3.1 the resolvent construction for the Dirichlet problem with C^τ coefficients and uniform strong ellipticity condition, we show that the principal symbol of a differential operator with uniform strong ellipticity condition is contained in a sector. We also introduce the parametrix construction of

$$\mathcal{A} = \begin{pmatrix} P_+ + G \\ T \end{pmatrix} : H^s(\mathbb{R}^n_+) \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix}$$

where P_+ is a truncated pseudodifferential operator, G is a singular Green operator and T is a trace operator in the case of non-smooth coefficients.

Definition 2.3.1. *A linear partial differential operator of order m on a bounded domain D in* $\mathbb{R}^n$ *is an expression of the form :*

$$A(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha = \sum_{k=0}^{m} \sum_{\alpha_1 + \alpha_2 + \cdots + \alpha_n = k} a_{\alpha_1 \cdots \alpha_n}(x) D_1^{\alpha_1} \cdots D_n^{\alpha_n} \tag{2.28}$$

where the complex matrix-valued coefficients $a_\alpha(x)$ are defined on D and $m \in \mathbb{N}$. The principal symbol of A will be denoted by $\sigma^m(A)(x, \xi)$ and

$$\sigma^m(A)(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha,$$

which is a homogeneous polynomial of degree m with respect to ξ. We call A elliptic, if for every $x \in D$ and every non-zero $\xi \in \mathbb{R}^n$

$$\sigma^m(A)(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$$

is invertible. The operator A is said to be uniform strong ellipticity if there exists a constant $c > 0$ such that for every $x \in D$ and $\xi \in \mathbb{R}^n$

$$\operatorname{Re} \sigma^{2m}(A)(x, \xi) \geq c \mid \xi \mid^{2m} . \tag{2.29}$$

The order $2m$ (even) in (2.29) is necessary, since by homogeneity for odd order $2m + 1$, we have $\operatorname{Re} \sigma^{2m+1}(A)(x, -\xi) = -\operatorname{Re} \sigma^{2m+1}(A)(x, \xi)$, so the principal symbol takes on both positive and negative values.

Remark 2.3.2. *Inequality (2.29) implies that*

$$\operatorname{Re} \sigma^{2m}(A)(x, \xi) > 0$$

for each $x \in \overline{D}$ and $\xi \neq 0$, i.e. ellipticity.

Theorem 2.3.3. *If the partial differential operator (2.28) has bounded coefficients and satisfies a uniform strong ellipticity condition, then the spectrum of principal symbol of A is contained in a sector*

$$\left\{z = z_1 + iz_2; z \neq 0, |\arg z| < \frac{\pi}{2} - \epsilon\right\}$$

for some $0 < \epsilon < \frac{\pi}{2}$.

Proof. Since A satisfies a uniform strong ellipticity condition we have

$$c \mid \xi \mid^{2m} \leq \operatorname{Re} \sigma^{2m}(A)(x, \xi) \leq C \mid \xi \mid^{2m} .$$

For all $v \in \mathbb{C}^n$, set $z = z_1 + iz_2 = \langle \sigma^{2m}(A)(x, \xi)v, v\rangle_{\mathbb{C}^n}$, then

$$\langle \sigma^{2m}(A)(x, \xi)v, v\rangle_{\mathbb{C}^n} = \operatorname{Im}\langle \sigma^{2m}(A)(x, \xi)v, v\rangle_{\mathbb{C}^n} + \operatorname{Re}\langle \sigma^{2m}(A)(x, \xi)v, v\rangle_{\mathbb{C}^n}.$$

Since

$$\begin{aligned} |\operatorname{Im}\langle \sigma^{2m}(A)(x, \xi)v, v\rangle_{\mathbb{C}^n}| &\leq |\langle \sigma^{2m}(A)(x, \xi)v, v\rangle_{\mathbb{C}^n}| \\ &\leq c'|\xi|^{2m}\langle v, v\rangle_{\mathbb{C}^n} \leq c'' \operatorname{Re}\langle \sigma^{2m}(A)(x, \xi)v, v\rangle_{\mathbb{C}^n}, \end{aligned}$$

we conclude that

$$\frac{|z_2|}{z_1} = \frac{|\operatorname{Im}\langle \sigma^{2m}(A)(x, \xi)v, v\rangle_{\mathbb{C}^n}|}{\operatorname{Re}\langle \sigma^{2m}(A)(x, \xi)v, v\rangle_{\mathbb{C}^n}} \leq c''.$$

Therefore $|\arg z| = |\arctan(\frac{z_2}{z_1})| \leq \arctan(c'')$, since z_1 is positive. □

Theorem 2.3.4. *a) Let*

$$\mathcal{A} = \begin{pmatrix} P_+ + G \\ T \end{pmatrix}$$

where P_+ is a truncated pseudodifferential operator of order zero to $\mathbb{R}^n_+$ satisfying the transmission condition at $x_n = 0$, G is a zero-order singular Green operator, such that $P_+ + G$ is of class $r \in \mathbb{Z}$, and T is a trace operator of order $\frac{1}{2}$ and class r, all with C^τ_*-smoothness in x. Then $\mathcal{A}$ maps continuously

$$\mathcal{A} = \begin{pmatrix} P_+ + G \\ T \end{pmatrix} : H^s(\mathbb{R}^n_+) \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix}$$

when

i) $|s| < \tau$,

ii) $|s - \frac{1}{2}| < \tau$,

iii) $s > r - \frac{1}{2}$*(class restriction).*

b) Let $\mathcal{A}$ be as in (a), and classical and uniformly elliptic. Then there exists an operator $\mathcal{B}^0$,

$$\mathcal{B}^0 = \begin{pmatrix} R^0 & K^0 \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to H^s(\mathbb{R}^n_+)$$

with R^0 of order 0 and class r (being the sum of a pseudodifferential operator and a singular Green operator), K^0 a Poisson operator of order $\frac{1}{2}$, with $C^{\tau-[\theta]}_*$-smoothness in x, satisfies that $\mathcal{B}^0$ is continuous for all s as in (a), and $\mathcal{R} = \mathcal{A}\mathcal{B}^0 - I$ is continuous

$$\mathcal{R} : \begin{matrix} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s-\theta-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix}$$

when

i) $-\tau + \theta < s < \tau - [\theta]$

ii) $s - \frac{1}{2} > -\tau + \theta$

iii) $s - \theta > r - \frac{1}{2}$*(class restriction)*

iv) $0 < \theta < \tau$.

Proof. Cf. [1, Theorem. A.8] and [1, Theorem 1.1, 1.2 and 6.4]. □

2.4 The Behavior of Pseudodifferential Operators with Non-Smooth Symbol at the Boundary

Let Q be a pseudodifferential operator. Hörmander showed in [11] that the maps

$$C^\infty_0(\mathbb{R}^{n-1}) \ni u \longrightarrow \lim_{x_n \to 0^+} D^l_n Q(u \otimes \delta^j)$$

are given by pseudodifferential operators. Here $\delta^j = D_n^j\delta$, δ denoting the Dirac measure.
Abels illustrated in [1] the effect of the transmission condition for non-smooth truncated pseudodifferential operators $r^+p(x, D_x)(u \otimes \delta)$. Therefore we will show Hörmander's formula for non-smooth symbols with the transmission property.

Lemma 2.4.1. *Let $u \in H^s(\mathbb{R}^{n-1})$, $\delta^j = D_n^j\delta$, δ denoting the Dirac measure. Then $u \otimes \delta^j \in H^t(\mathbb{R}^n)$ whenever $t + j < -\frac{1}{2}$ and $t + j + \frac{1}{2} \leq s$.*

Proof.

$$\begin{aligned} \|u \otimes \delta^j\|_t^2 &= \int |\hat{u}(\xi')|^2 \int \langle \xi' \rangle^{2t} (\xi_n)^{2j} đ\xi_n đ\xi' \\ &\overbrace{\leq}^{t+j<-\frac{1}{2}} C \int |\hat{u}(\xi')|^2 \langle \xi' \rangle^{2t+2j+1} đ\xi' \\ &\overbrace{\leq}^{t+j+\frac{1}{2}\leq s} C\|u\|_s \end{aligned}$$

□

Theorem 2.4.2. *Let Q be a pseudodifferential operator of order $m \in \mathbb{Z}$ such that its symbol $q \in C_*^\tau S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$ has the transmission property. Then, for every $u \in C_0^\infty(\mathbb{R}^{n-1})$, the boundary values*

$$Q^l u = \gamma_l r^+ Q(u \otimes \delta), \; l < \tau, \; l \in \mathbb{N}_0 \;,$$

are given by pseudodifferential operators Q^l with symbols

$$q_l(x', \xi') = \lim_{x_n \to 0^+} r^+ \mathcal{F}^{-1}_{\xi_n \to x_n} h^+[(D_{x_n} + \xi_n)^l q(x, \xi)].$$

Proof. From Lemma 2.2.2

$$r^+q(x, D_x)(u \otimes \delta) = r^+\mathcal{F}^{-1}_{\xi_n \to x_n}(h^+q(x, \xi)) \in C_*^\tau S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n; S(\overline{\mathbb{R}}_+)) \tag{2.30}$$

for every $u \in C_0^\infty(\mathbb{R}^{n-1})$. Now we apply the differential operator $D_{x_n}^l$ ($l < \tau$) and we obtain

$$D_{x_n}^l r^+q(x, D_x)(u \otimes \delta) = (2\pi)^{-n} \int e^{ix'\xi'} \hat{u}(\xi') r^+ \int e^{ix_n\xi_n} \Big((D_{x_n} + \xi_n)^l q(x, \xi) \Big) d\xi_n d\xi',$$

here the integral in ξ_n has to be interpreted as an oscillatory integral. Hence

$$\lim_{x_n \to 0^+} D_{x_n}^l r^+ q(x, D_x)(u \otimes \delta) = (2\pi)^{1-n} \int e^{ix'\xi'} \hat{u}(\xi') q_l(x', \xi') d\xi'$$

where

$$\begin{aligned} q_l(x', \xi') &= (2\pi)^{-1} \lim_{x_n \to 0^+} r^+ \int e^{ix_n\xi_n} (D_{x_n} + \xi_n)^l q(x, \xi) d\xi_n \\ &= \lim_{x_n \to 0^+} r^+ \mathcal{F}^{-1}_{\xi_n \to x_n} h^+[(D_{x_n} + \xi_n)^l q(x, \xi)] \\ &\in C_*^{\tau - l} S_{1,0}^{m+l}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}), \end{aligned}$$

by (2.30). It follows that the operator $\lim_{x_n \to 0^+} D_{x_n}^l r^+ Q(u \otimes \delta)$ is a pseudodifferential operator. □

Corollary 2.4.3. *If in the Theorem 2.4.2 each term q_j in the asymptotic expansion $q \sim \sum_j q_j$ is a rational function of ξ_n, then the symbols can be computed as in Hörmander [10,* Theorem 2.1.4*] and equation* (1.4) *with $\Gamma_{\xi'}$. Here $\Gamma_{\xi'}$ is a contour in $\mathbb{C}$, which has the poles of q^j in the upper half plane in its interior.*

2.5 Parametrices to Certain Hypoelliptic Operators

We start with the introduction of a notion of parameter-dependent hypoellipticity condition for non-smooth symbols. We will then construct a parametrix for pseudodifferential operators with symbols in $C^\tau_* S^m_{1,\delta}$ for $0 \leq \delta < 1$. For the Hörmander classes $S^m_{\rho,\delta}$, this condition has been introduced by Hörmander in [12, Theorem 4.2]. We follow here the approach developed by O. Bilyj, E. Schrohe and J. Seiler in [5]. In [15], Marschall has constructed the parametrix for elliptic operators with coefficients in $C^\tau_* S^m_{1,0}$.

Definition 2.5.1. *Let $p \in C^\tau_* S^m_{1,\delta}$ for some $m \geq 0$, $\tau \in (0, \infty)$, and $0 \leq \delta < 1$. Consider the following sector:*

$$S(\epsilon) = \left\{\lambda = r^2 e^{i\eta}, \epsilon \leq \eta \leq 2\pi - \epsilon\right\} \tag{2.31}$$

for some $0 < \epsilon < \pi$. We say that $p(x, \xi)$ is hypoelliptic, if there exist constants $C, c > 0$ such that for $x, \xi \in \mathbb{R}^n$, $|\xi| \geq C$, the resolvent set of $p(x, \xi)$ contains

$$S(\epsilon) \cup \{|\lambda| \leq c\}$$

and for $\lambda \in S(\epsilon) \cup \{|\lambda| \leq c\}$, $\alpha \in \mathbb{N}_0$ and $|\beta| \leq [\tau]$,

$$|\partial_x^\beta D_\xi^\alpha p(x, \xi)\, (p(x, \xi) - \lambda)^{-1}\,| \leq c_{\alpha,\beta} \langle\xi\rangle^{-|\alpha|+\delta'|\beta|} \tag{2.32}$$

$$|\,(p(x, \xi) - \lambda)^{-1}\, \partial_x^\beta D_\xi^\alpha p(x, \xi)| \leq c_{\alpha,\beta} \langle\xi\rangle^{-|\alpha|+\delta'|\beta|} \tag{2.33}$$

$$\begin{aligned} \|\partial_x^\beta D_\xi^\alpha p(\cdot, \xi)\, (p(\cdot, \xi) - \lambda)^{-1}\,\|_{C_*^{\tau-|\beta|}} &\leq c_{\alpha,\beta} \langle\xi\rangle^{-|\alpha|+\delta'|\beta|+\delta'(\tau-|\beta|)} \\ &= c_{\alpha,\beta} \langle\xi\rangle^{-|\alpha|+\delta'\tau} \end{aligned} \tag{2.34}$$

$$\begin{aligned} \|\,(p(\cdot, \xi) - \lambda)^{-1}\, \partial_x^\beta D_\xi^\alpha p(\cdot, \xi)\|_{C_*^{\tau-|\beta|}} &\leq c_{\alpha,\beta} \langle\xi\rangle^{-|\alpha|+\delta'|\beta|+\delta'(\tau-|\beta|)} \\ &= c_{\alpha,\beta} \langle\xi\rangle^{-|\alpha|+\delta'\tau} \end{aligned} \tag{2.35}$$

for some $0 \leq \delta \leq \delta' < 1$.

Clearly, for $|\xi| \geq C$, $p(x, \xi)$ is invertible and the spectrum of $p(x, \xi)$ is a subset of $\{\lambda \in \mathbb{C} \setminus S(\epsilon); |\lambda| < c'|p(x, \xi)|\}$, for any $c' > 1$. Additionally, since $(p - \lambda)^{-1} = -\lambda^{-1}(1 - p(p - \lambda)^{-1})$ and 0 is not in the spectrum of $p(x, \xi)$, then outside the neighborhood of $\lambda = 0$,

$$|(p(x, \xi) - \lambda)^{-1}| = |\lambda|^{-1}|1 - p(x, \xi)(p(x, \xi) - \lambda)^{-1}| \leq c'_{0,0}\langle\lambda\rangle^{-1}, \tag{2.36}$$

$$\begin{aligned} \|(p(\cdot, \xi)-\lambda)^{-1}\|_{C^\tau_*} &= |\lambda|^{-1}\|1-p(\cdot, \xi)(p(\cdot, \xi)-\lambda)^{-1}\|_{C^\tau_*} \leq |\lambda|^{-1}\left(1+\|p(\cdot, \xi)(p(\cdot, \xi)-\lambda)^{-1}\|_{C^\tau_*}\right) \\ &\leq |\lambda|^{-1}\left(1 + c_0\langle\xi\rangle^{\delta'\tau}\right) \leq c'_0\langle\lambda\rangle^{-1}\langle\xi\rangle^{\delta'\tau},\ |\xi| \geq C. \end{aligned}$$

Now we want to construct a parameter-dependent parametrix to $p - \lambda$.

Remark 2.5.2. *Let* $p \in C_*^\tau S_{1,\delta}^m$. *If* $|\xi| < C$, *then* $|p(x,\xi)| \le c_0$. *We can conclude for* $|\lambda| > c_0$ *and* $|\xi| < C$*:*

$$|p(x,\xi)| \le c_0 < |\lambda|.$$

This shows that $p(x,\xi) - \lambda$ *is invertible for* $|\lambda| > c_0$ *and* $|\xi| < C$.

Definition 2.5.3. *Let* $q_0(x,\xi,\lambda) = q_0^l(x,\xi,\lambda) = q_0^r(x,\xi,\lambda) = (p(x,\xi) - \lambda)^{-1}$, *for* $|\xi| \ge C$ *and* $\lambda \in S(\epsilon)$, *or* $|\xi| < C$ *and* $|\lambda| > c_0$. *Define for* $j = 1, 2, ..., [\theta]$, $\theta \notin \mathbb{N}$, $\theta < \tau$*:*

a) (Right parametrix)

$$q_j^r(x,\xi,\lambda) = -q_0(x,\xi,\lambda) \sum_{i=1}^{j} \sum_{|\alpha|=i} \frac{1}{\alpha!} D_\xi^\alpha p(x,\xi) \partial_x^\alpha q_{j-i}^r(x,\xi,\lambda).$$

b) (Left parametrix)

$$q_j^l(x,\xi,\lambda) = -\sum_{i=1}^{j} \sum_{|\alpha|=i} \frac{1}{\alpha!} D_\xi^\alpha q_{j-i}^l(x,\xi,\lambda) \partial_x^\alpha p(x,\xi)\, q_0(x,\xi,\lambda)\,.$$

Note that for an arbitrary derivative $\partial = \partial_{x_j}$ or $\partial = \partial_{\xi_j}$, we have

$$\partial(p(x,\xi) - \lambda)^{-1} = -(p(x,\xi) - \lambda)^{-1} \partial p(x,\xi)(p(x,\xi) - \lambda)^{-1}.$$

Theorem 2.5.4. *Let* $p \in C_*^\tau S_{1,\delta}^m$ *satisfy the assumption of Definition* (2.5.1) *for some* δ' *such that* $0 \le \delta \le \delta' < 1$.

a) Then $\lambda \mapsto \langle\lambda\rangle q_j^l(\lambda)$, $\lambda \mapsto \langle\lambda\rangle q_j^r(\lambda)$ *are continuous and bounded from* $\{\lambda \in S(\epsilon); |\lambda| > c_0\}$ *to* $C_*^{\tau-j} S_{1,\delta'}^{j(\delta'-1)}$ *for* $j = 0, 1, ..., [\theta], \theta \notin \mathbb{N}, \theta < \tau$.

b) Let $q^r(x,\xi,\lambda) = \sum_{j=0}^{[\theta]} q_j^r(x,\xi,\lambda)$ *and* $q^l(x,\xi,\lambda) = \sum_{j=0}^{[\theta]} q_j^l(x,\xi,\lambda)$. *Then* q^r, q^l *are right and left parametrices for p such that*

$$\mathcal{R}^r(x,D_x,\lambda) = (p(x,D_x) - \lambda) q^r(x,D_x,\lambda) - I : H^{s+m-\theta(1-\delta')}(\mathbb{R}^n) \to H^s(\mathbb{R}^n),$$

for

$$\begin{cases} -(\tau - [\theta])(1-\delta) & < \quad s < \tau - [\theta], \\ -(1-\delta')(\tau - [\theta] - \theta) & < \quad s + m < \tau - [\theta], \\ -(1-\delta')(\tau - [\theta] - \theta) & < \quad s \\ 0 & < \quad \theta < \tau - [\theta] \end{cases}$$

$$\mathcal{R}^l(x,D_x,\lambda) = q^l(x,D_x,\lambda)(p(x,D_x) - \lambda) - I : H^{s+m-\theta(1-\delta')}(\mathbb{R}^n) \to H^s(\mathbb{R}^n),$$

for

$$\begin{cases} -(\tau - [\theta])(1-\delta') & < \quad s < \tau - [\theta], \\ -(1-\delta)(\tau - \theta) & < \quad s < \tau, \\ 0 & < \quad \theta < \tau - [\theta] \end{cases}$$

are bounded. The norms of $\mathcal{R}^r(x,D_x,\lambda)$ *and* $\mathcal{R}^l(x,D_x,\lambda)$ *are* $O(\langle\lambda\rangle^{-1})$ *in* $S(\epsilon)$.

Proof. a) For $|\alpha_1|+|\alpha_2|+...+|\alpha_l| = |\alpha|+j$, $|\beta_1|+|\beta_2|+...+|\beta_l| = |\beta|+j \leq [\tau]$, $l \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^n$, $D_\xi^\alpha \partial_x^\beta q_j^r$ is a linear combination of terms

$$q_0(x,\xi,\lambda)D_\xi^{\alpha_1}\partial_x^{\beta_1}p(x,\xi)q_0(x,\xi,\lambda)...D_\xi^{\alpha_l}\partial_x^{\beta_l}p(x,\xi)q_0(x,\xi,\lambda).$$

To simplify the following statement write $D_\xi^{\alpha_0}\partial_x^{\beta_0}p(x,\xi) = 1$. Then from Remark 2.1.8 and (2.32)-(2.36)

$$\begin{aligned} \|D_\xi^\alpha\partial_x^\beta q_j^r(x,\xi,\lambda)\|_{C_*^{\tau-|\beta|}} &= \|q_0(x,\xi,\lambda)D_\xi^{\alpha_1}\partial_x^{\beta_1}p(x,\xi)q_0(x,\xi,\lambda)...D_\xi^{\alpha_l}\partial_x^{\beta_l}p(x,\xi)q_0(x,\xi,\lambda)\|_{C_*^{\tau-|\beta|}} \\ &\leq \sum_{j=0}^{l}\|D_\xi^{\alpha_j}\partial_x^{\beta_j}p(x,\xi)q_0(x,\xi,\lambda)\|_{C_*^{\tau-|\beta|}}\left(\Pi_{\substack{i=0\\i\neq j}}^{l}\|D_\xi^{\alpha_i}\partial_x^{\beta_i}p(x,\xi)q_0(x,\xi,\lambda)\|_{L^\infty}\right) \\ &\leq c'_{\alpha,\beta}\langle\lambda\rangle^{-1}\langle\xi\rangle^{-|\alpha|-j+\delta'(\tau-|\beta|-j)+\delta'(|\beta|+j)}. \end{aligned}$$

Therefore $\langle\lambda\rangle q_j^r$ is continuous and bounded. Similarly we can verify a) for $\langle\lambda\rangle q_j^l$.

b) We define the remainders $\mathcal{R}_j^r$ and $\mathcal{R}_j^l$ for $j = 0,1,...,[\theta]$ by using Theorem 2.1.18 as follows:

$$\begin{aligned} (p(x,D_x)-\lambda)q_0^r(x,D_x,\lambda)-I &= \sum_{1\leq|\alpha|\leq[\theta]}\frac{1}{\alpha!}Op(D_\xi^\alpha p(x,\xi)\partial_x^\alpha q_0^r(x,\xi,\lambda)) \\ &\quad + \mathcal{R}_0^r(x,D_x,\lambda) \\ (p(x,D_x)-\lambda)q_1^r(x,D_x,\lambda) &= -\sum_{|\alpha|=1}Op(D_\xi^\alpha p(x,\xi)\partial_x^\alpha q_0^r(x,\xi,\lambda)) \\ &\quad + \sum_{1\leq|\alpha|\leq[\theta]-1}\frac{1}{\alpha!}Op(D_\xi^\alpha p(x,\xi)\partial_x^\alpha q_1^r(x,\xi,\lambda)) \\ &\quad + \mathcal{R}_1^r(x,D_x,\lambda) \\ (p(x,D_x)-\lambda)q_j^r(x,D_x,\lambda) &= -\sum_{i=1}^{j}\sum_{|\alpha|=i}\frac{1}{\alpha!}Op(D_\xi^\alpha p(x,\xi)\partial_x^\alpha q_{j-i}^r(x,\xi,\lambda)) \\ &\quad + \sum_{1\leq|\alpha|\leq[\theta]-j}\frac{1}{\alpha!}Op(D_\xi^\alpha p(x,\xi)\partial_x^\alpha q_j^r(x,\xi,\lambda)) \\ &\quad + \mathcal{R}_j^r(x,D_x,\lambda) \end{aligned}$$

such that $\mathcal{R}_j^r(x,D_x,\lambda) : H^{s+m+j(\delta'-1)-\theta(1-\delta')}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$, for

$$\begin{cases} -(\tau-j)(1-\delta) < s < \tau-j, \\ -(1-\delta')(\tau-j-\theta) < s+m < \tau-j, \\ -(1-\delta')(\tau-j-\theta) < s, \\ 0 < \theta < \tau-j \end{cases}$$

are bounded operators and

$$\lambda \mapsto \langle\lambda\rangle\|\mathcal{R}_j^r\|_{\mathcal{L}(H^{s+m+j(\delta'-1)-\theta(1-\delta')},H^s)} \tag{2.37}$$

is bounded.

$$\begin{aligned} q_0^l(x,D_x,\lambda)(p(x,D_x)-\lambda)-I &= \sum_{1\le|\alpha|\le[\theta]} \frac{1}{\alpha!} Op(D_\xi^\alpha q_0^l(x,\xi,\lambda)\partial_x^\alpha p(x,\xi)) \\ &\quad + \mathcal{R}_0^l(x,D_x,\lambda) \\ q_1^l(x,D_x,\lambda)(p(x,D_x)-\lambda) &= -\sum_{|\alpha|=1} Op(D_\xi^\alpha q_0^l(x,\xi,\lambda)\partial_x^\alpha p(x,\xi)) \\ &\quad + \sum_{1\le|\alpha|\le[\theta]-1} \frac{1}{\alpha!} Op(D_\xi^\alpha q_1^l(x,\xi,\lambda)\partial_x^\alpha p(x,\xi)) \\ &\quad + \mathcal{R}_1^l(x,D_x,\lambda) \end{aligned}$$

$$\begin{aligned} q_j^l(x,D_x,\lambda)(p(x,D_x)-\lambda) &= -\sum_{i=1}^{j}\sum_{|\alpha|=i} \frac{1}{\alpha!} Op(D_\xi^\alpha q_{j-i}^l(x,\xi,\lambda)\partial_x^\alpha p(x,\xi)) \\ &\quad + \sum_{1\le|\alpha|\le[\theta]-j} \frac{1}{\alpha!} Op(D_\xi^\alpha q_j^l(x,\xi,\lambda)\partial_x^\alpha p(x,\xi)) \\ &\quad + \mathcal{R}_j^l(x,D_x,\lambda), \end{aligned}$$

such that $\mathcal{R}_j^l(x,D_x,\lambda) : H^{s+m+j(\delta'-1)-\theta(1-\delta')}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$, for

$$\begin{cases} -(\tau-j)(1-\delta') < s < \tau-j, \\ -(1-\delta)(\tau-\theta) < s < \tau, \\ 0 < \theta < \tau-j \end{cases}$$

are bounded operators and

$$\lambda \mapsto \langle\lambda\rangle \|\mathcal{R}_j^l\|_{\mathcal{L}(H^{s+m+j(\delta'-1)-\theta(1-\delta')},H^s)} \tag{2.38}$$

is bounded. Therefore

$$\begin{aligned} (p-\lambda)\sharp_{[\theta]} q_0^r &= I + \sum_{1\le|\alpha|\le[\theta]} \frac{1}{\alpha!} D_\xi^\alpha p(x,\xi)\partial_x^\alpha q_0^r(x,\xi,\lambda), \\ (p-\lambda)\sharp_{[\theta]-1} q_1^r &= -D_\xi p(x,\xi)\partial_x q_0^r(x,\xi,\lambda) + \\ &\quad + \sum_{1\le|\alpha|\le[\theta]-1} \frac{1}{\alpha!} D_\xi^\alpha p(x,\xi)\partial_x^\alpha q_1^r(x,\xi,\lambda), \\ (p-\lambda)\sharp_{[\theta]-2}\ q_2^r &= -D_\xi p(x,\xi)\partial_x q_1^r(x,\xi,\lambda) \\ &\quad - \sum_{|\alpha|=2} \frac{1}{\alpha!} D_\xi^\alpha p(x,\xi)\partial_x^\alpha q_0^r(x,\xi,\lambda) \\ &\quad + \sum_{1\le|\alpha|\le[\theta]-2} \frac{1}{\alpha!} D_\xi^\alpha p(x,\xi)\partial_x^\alpha q_2^r(x,\xi,\lambda), \\ &\quad \ldots. \end{aligned}$$

Since $q^r(x,\xi,\lambda) = \sum_{j=0}^{[\theta]} q_j^r(x,\xi,\lambda)$ and $q^l(x,\xi,\lambda) = \sum_{j=0}^{[\theta]} q_j^l(x,\xi,\lambda)$, then:

$$\begin{aligned}
\mathcal{R}^r(p-\lambda, q^r) &= \sum_{j=0}^{[\theta]} \mathcal{R}^r_j(x, D_x, \lambda) \\
&= (p(x,D_x)-\lambda)q^r(x,D_x,\lambda) - (\sum_{j=0}^{[\theta]} (p-\lambda)\sharp_{[\theta]-j} q^r_j)(x,D_x,\lambda) \\
&= (p(x,D_x)-\lambda)q^r(x,D_x,\lambda) - I : H^{s+m-\theta(1-\delta')}(\mathbb{R}^n) \to H^s(\mathbb{R}^n),
\end{aligned}$$

for

$$\begin{cases} -(\tau-[\theta])(1-\delta) &< s < \tau-[\theta], \\ -(1-\delta')(\tau-[\theta]-\theta) &< s+m < \tau-[\theta], \\ -(1-\delta')(\tau-[\theta]-\theta) &< s \\ 0 &< \theta < \tau-[\theta] \end{cases}$$

is bounded and

$$\lambda \mapsto \langle\lambda\rangle \|\mathcal{R}^r\|_{\mathcal{L}(H^{s+m-\theta(1-\delta')}, H^s)} \tag{2.39}$$

is continuous.

$$\begin{aligned}
\mathcal{R}^l(q^l, p-\lambda) &= \sum_{j=0}^{[\theta]} \mathcal{R}^l_j(x, D_x, \lambda) \\
&= q^l(x,D_x,\lambda)(p(x,D_x)-\lambda) - (\sum_{j=0}^{[\theta]} q^l_j \sharp_{[\theta]-j}(p-\lambda))(x,D_x,\lambda) \\
&= q^l(x,D_x,\lambda)(p(x,D_x)-\lambda) - I : H^{s+m-\theta(1-\delta')}(\mathbb{R}^n) \to H^s(\mathbb{R}^n),
\end{aligned}$$

for

$$\begin{cases} -(\tau-[\theta])(1-\delta') &< s < \tau-[\theta], \\ -(1-\delta)(\tau-\theta) &< s < \tau, \\ 0 &< \theta < \tau-[\theta] \end{cases}$$

is bounded and

$$\lambda \mapsto \langle\lambda\rangle \|\mathcal{R}^l\|_{\mathcal{L}(H^{s+m-\theta(1-\delta')}, H^s)} \tag{2.40}$$

is continuous.

□

Corollary 2.5.5. *Let $m-\theta(1-\delta') \leq 0$. From* (2.39) *and* (2.40) *$\langle\lambda\rangle\mathcal{R}^r$ and $\langle\lambda\rangle\mathcal{R}^l$ are bounded. If $|\lambda| \to \infty$, then $\mathcal{R}^r$ and $\mathcal{R}^l$ tend to zero and we conclude with a Neumann series argument for large $|\lambda| \geq R$, $1+\mathcal{R}^r(\lambda)$ and $1+\mathcal{R}^l(\lambda)$ are invertible.*

Chapter 3

The Resolvent Construction for the Dirichlet Problem in the Case of Non-Smooth Coefficients

3.1 Dirichlet Resolvent in $\mathbb{R}^n_+$

In [2] the resolvent $(A^D_{\gamma_0} - \lambda)^{-1}$ and the Poisson operator $K^\lambda_{\gamma_0}$ for the Dirichlet problem in the non-smooth situation are constructed for $0 < \tau \le 1$ on non-smooth domains. In this section we generalize this result to all $\tau > 0$ on compact manifold.

Let

$$A(x, D) = \sum_{i,j=1}^{n} a^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b^i(x)\partial_{x_i} + c(x) \tag{3.1}$$

be a second-order uniformly strongly elliptic differential operator with real coefficients. Fix $\tau > 0$. We assume:

(1) $a^{ij} \in C^\tau(\overline{\mathbb{R}^n_+})$ and there exists a constant $c_0 > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \ge c_0|\xi|^2, x \in \overline{\mathbb{R}^n_+}, \xi \in \mathbb{R}^n \tag{3.2}$$

(2) $b^i \in C^\tau(\overline{\mathbb{R}^n_+})$

(3) $c \in C^\tau(\overline{\mathbb{R}^n_+})$ and $c \le 0$ on $\overline{\mathbb{R}^n_+}$.

Definition 3.1.1. *Let $s > -\frac{3}{2}$. The Dirichlet realization of A in $H^s(\mathbb{R}^n_+)$ is the unbounded operator $A^D_{\gamma_0}$ with domain $D(A^D_{\gamma_0}) = \left\{u \in H^{s+2}(\mathbb{R}^n_+); \gamma_0 u = 0\right\}$.*

Theorem 3.1.2. *Let A and τ be as above. Then For $\lambda \in \rho(A^D_{\gamma_0})$, $-\tau + \frac{1}{2} < s < \tau$ and $s > -\frac{3}{2}$, the operator*

$$\begin{pmatrix} A-\lambda \\ \gamma_0 \end{pmatrix} : H^{s+2}(\mathbb{R}^n_+) \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix} \quad ;$$

has an inverse

$$\begin{pmatrix} R^D(\lambda) & K^D(\lambda) \end{pmatrix} = \begin{pmatrix} (A^D_{\gamma_0} - \lambda)^{-1} & K^\lambda_{\gamma_0} \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to H^{s+2}(\mathbb{R}^n_+).$$

On the rays $\lambda = re^{i\eta}$ with $\eta \in (\frac{\pi}{2} - \epsilon, \frac{3\pi}{2} + \epsilon)$ for $0 < \epsilon < \frac{\pi}{2}$ (Outside the range of the principal symbol), the inverse exists for $|\lambda|$ sufficiently large. The operators $R^D(\lambda)$ and $K^D(\lambda)$ have the structure in (5.9) *and satisfy estimates* (5.10), (5.11) *and* (5.12) .

Proof. Step 1. For $\lambda = 0$ we write $\mathcal{A}^D = \begin{pmatrix} A \\ \gamma_0 \end{pmatrix}$.

$$\mathcal{A}^D : H^{s+2}(\mathbb{R}^n_+) \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix} \quad ,$$

is continuous for all $-\tau < s < \tau$ and $s > -\frac{3}{2}$. If we use the order-reducing operators of Remark 2.2.3 to reduce the orders to zero, then we have by an application of Theorem 2.3.4:

$$\mathcal{A}^D_1 = \begin{pmatrix} I & 0 \\ 0 & \Lambda^2_0 \end{pmatrix} \mathcal{A}^D \Lambda^{-2}_{-,\mathbb{R}_+} = \begin{pmatrix} A\Lambda^{-2}_{-,\mathbb{R}_+} \\ \Lambda^2_0 \gamma_0 \Lambda^{-2}_{-,\mathbb{R}_+} \end{pmatrix} : H^s(\mathbb{R}^n_+) \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \quad .$$

$A\Lambda^{-2}_{-,\mathbb{R}_+}$ is of order zero and $\Lambda^2_0\gamma_0\Lambda^{-2}_{-,\mathbb{R}_+}$ is of order $\frac{1}{2}$ and class -1. $\mathcal{A}^D_1$ is continuous for all $-\tau < s < \tau$ with $s > -\frac{3}{2}$. (We do not need the restriction $|s - \frac{1}{2}| < \tau$ here, since γ_0 is smooth and it does not have any coefficient in C^τ.)
By Theorem 2.3.4 $\mathcal{A}^D_1$ has a parametrix $B^{0,D}_1$ of order zero and class -1,

$$\mathcal{B}^{0,D}_1 = \begin{pmatrix} R^{0,D}_1 & K^{0,D}_1 \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to H^s(\mathbb{R}^n_+).$$

Here $R^{0,D}_1$ is of order zero, and $K^{0,D}_1$ is a Poisson operator of order $\frac{1}{2}$, having symbol-kernel in $C^\tau S^{-\frac{1}{2}}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, S(\overline{\mathbb{R}_+}))$. Since $K^{0,D}_1$ has coefficients in C^τ, we have an extra restriction $|s - \frac{1}{2}| < \tau$ on s. Therefore $\mathcal{B}^{0,D}_1$ is continuous for $-\tau + \frac{1}{2} < s < \tau$ and $s > -\frac{3}{2}$. The remainder $\mathcal{R}^D_1 = \mathcal{A}^D_1\mathcal{B}^{0,D}_1 - I$ satisfies

$$\mathcal{R}^D_1 = \begin{pmatrix} A\Lambda^{-2}_{-,\mathbb{R}_+} \\ \Lambda^2_0 \gamma_0 \Lambda^{-2}_{-,\mathbb{R}_+} \end{pmatrix} \begin{pmatrix} R^{0,D}_1 & K^{0,D}_1 \end{pmatrix} - I = \begin{matrix} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s-\theta-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix}$$

when $0 < \theta < \tau$, $-\tau + \frac{1}{2} + \theta < s < \tau - [\theta]$ and $s > -\frac{3}{2} + \theta$.
We obtain

$$\mathcal{A}_1^D \mathcal{B}_1^{0,D} = I + \mathcal{R}_1^D \text{ or } \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix} \mathcal{A}^D \Lambda_{-,\mathbb{R}_+}^{-2} \mathcal{B}_1^{0,D} = I + \mathcal{R}_1^D.$$

Therefore

$$\mathcal{A}^D \Lambda_{-,\mathbb{R}_+}^{-2} \mathcal{B}_1^{0,D} \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix} = I + \mathcal{R}'^D, \text{ with } \mathcal{R}'^D = \begin{pmatrix} \mathcal{R}'^D_1 \\ \mathcal{R}'^D_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{-2} \end{pmatrix} \mathcal{R}_1^D \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix}.$$

Step 2. Set

$$\mathcal{B}_0^D = \Lambda_{-,\mathbb{R}_+}^{-2} \mathcal{B}_1^{0,D} \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix} = \begin{pmatrix} \Lambda_{-,\mathbb{R}_+}^{-2} R_1^{0,D} & \Lambda_{-,\mathbb{R}_+}^{-2} K_1^{0,D} \Lambda_0^2 \end{pmatrix} = \begin{pmatrix} R_0^D & K_0^D \end{pmatrix}.$$

It is a parametrix of $\mathcal{A}^D$, and

$$\mathcal{A}^D \mathcal{B}_0^D = I + \mathcal{R}'^D.$$

$$\mathcal{B}_0^D : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to H^{s+2}(\mathbb{R}^n_+),$$

is continuous when

$$-\tau + \frac{1}{2} < s < \tau \text{ and } s > -\frac{3}{2}. \tag{3.3}$$

$$\mathcal{R}'^D = \begin{pmatrix} \mathcal{R}'^D_1 \\ \mathcal{R}'^D_2 \end{pmatrix} : \begin{matrix} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{3}{2}-\theta}(\mathbb{R}^{n-1}) \end{matrix} \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}), \end{matrix}$$

is continuous when

$$0 < \theta < \tau\,, \ -\tau + \frac{1}{2} + \theta < s < \tau - [\theta] \text{ and } s > -\frac{3}{2} + \theta. \tag{3.4}$$

The inequality (3.2) implies that the principal symbol of A takes its values in $\overline{\mathbb{R}}_-$. Now we replace A by $A-\lambda_0$ for a large $|\lambda_0|$ and the parameter $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$ by $\langle\xi\rangle = (1 + |\lambda_0| + |\xi|^2)^{\frac{1}{2}}$ in the order reducing operators. We first prove the theorem for some large λ_0 and then extend it to all $\lambda \in \rho(A^D_{\gamma_0})$.
The parametrix and remainder map as follows:

$$\mathcal{B}_0^D(\lambda_0) = \begin{pmatrix} R_0^D(\lambda_0) & K_0^D(\lambda_0) \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to H^{s+2}(\mathbb{R}^n_+), \tag{3.5}$$

$$\mathcal{R}^D(\lambda_0) : \begin{matrix} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{3}{2}-\theta}(\mathbb{R}^{n-1}) \end{matrix} \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix} \tag{3.6}$$

for s and θ as in (3.3) and (3.4).
In order to get the exact inverses, we apply the technique developed by Abels, Grubb and Wood in [2], which can also be invoked here. Details are given in the appendix. □

From Theorem 3.1.2 we easily obtain the structure of the resolvent for compact manifolds with boundary. We state it for the sake of completeness.

Let $\overline{D}$ be a smooth compact manifold with boundary. We have the operator A as in (3.1). Consider the situation, where $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1}$ are replaced by $\overline{D}$ and ∂D.

Theorem 3.1.3. *For $\lambda \in \rho(A^D_{\gamma_0})$, $-\tau + \frac{1}{2} < s < \tau$ and $s > -\frac{3}{2}$, the operator*

$$\begin{pmatrix} A-\lambda \\ \gamma_0 \end{pmatrix} : H^{s+2}(\overline{D}) \to \begin{matrix} H^s(\overline{D}) \\ \times \\ H^{s+\frac{3}{2}}(\partial D) \end{matrix} \quad ;$$

has an inverse

$$\begin{pmatrix} R^D(\lambda) & K^D(\lambda) \end{pmatrix} = \begin{pmatrix} (A^D_{\gamma_0}-\lambda)^{-1} & K^\lambda_{\gamma_0} \end{pmatrix} : \begin{matrix} H^s(\overline{D}) \\ \times \\ H^{s+\frac{3}{2}}(\partial D) \end{matrix} \to H^{s+2}(\overline{D}).$$

On the rays $\lambda = re^{i\eta}$ with $\eta \in (\frac{\pi}{2}-\epsilon, \frac{3\pi}{2}+\epsilon)$ for $0 < \epsilon < \frac{\pi}{2}$ (Outside the range of the principal symbol), the inverse exists for $|\lambda|$ sufficiently large. The operators $R^D(\lambda)$ and $K^D(\lambda)$ have the structure in (5.9) *and satisfy estimates* (5.10), (5.11), (5.12).

Proof. We cover the manifold with finitely many charts. We may assume that they are chosen in such a way that either the chart does not intersect the boundary ("interior charts") or else is of the form $X_j \times [0,1)$ where $X_j \subseteq \partial D$ ("boundary chart"). In each of these, construct a parameter-dependent parametrix to the local representation of $(A-\lambda, \gamma_0)$ (in interior charts use a parameter-dependent parametrix to $A-\lambda$ only). With the help of partition of unity these can be patched to global parametrix. The estimate follows from the estimates for the local parametrices. □

3.2 Neumann Resolvent in $\mathbb{R}^n_+$

In this section we construct the Neumann resolvent for operators with C^τ coefficients.

Definition 3.2.1. *Let $s > -\frac{1}{2}$. The Neumann realization of A in $H^s(\mathbb{R}^n_+)$ is the unbounded operator $A^N_{\gamma_1}$ with domain $D(A^N_{\gamma_1}) = \left\{u \in H^{s+2}(\mathbb{R}^n_+); \gamma_1 u = 0\right\}$.*

Theorem 3.2.2. *Let A be as* (3.1) *satisfying conditions (1), (2) and (3). Then for $\lambda \in \rho(A^N_{\gamma_1})$, $-\tau + \frac{1}{2} < s < \tau$ and $s > -\frac{1}{2}$, the operator*

$$\begin{pmatrix} A-\lambda \\ \gamma_1 \end{pmatrix} : H^{s+2}(\mathbb{R}^n_+) \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix}$$

has an inverse

$$\begin{pmatrix} R^N(\lambda) & K^N(\lambda) \end{pmatrix} = \begin{pmatrix} (A^N_{\gamma_1}-\lambda)^{-1} & K^\lambda_{\gamma_1} \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to H^{s+2}(\mathbb{R}^n_+).$$

On the rays $\lambda = re^{i\eta}$ with $\eta \in (\frac{\pi}{2} - \epsilon, \frac{3\pi}{2} + \epsilon)$ for $0 < \epsilon < \frac{\pi}{2}$ (Outside the range of the principal symbol), the inverse exists for $|\lambda|$ sufficiently large. The operators $R^N(\lambda)$ and $K^N(\lambda)$ have the structure in (3.19) *and satisfy estimates* (3.20)-(3.24) *.*

Proof. Step 1. For $\lambda = 0$ we write $\mathcal{A}^N = \begin{pmatrix} A \\ \gamma_1 \end{pmatrix}$.

$$\mathcal{A}^N : H^{s+2}(\mathbb{R}^n_+) \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} ,$$

is continuous for all $-\tau < s < \tau$ and $s > -\frac{1}{2}$. If we use the order-reducing operators of Remark 2.2.3 to reduce the orders to zero, then we have by an application of Theorem 2.3.4:

$$\mathcal{A}_1^N = \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^1 \end{pmatrix} \mathcal{A}^N \Lambda^{-2}_{-,\mathbb{R}_+} = \begin{pmatrix} A\Lambda^{-2}_{-,\mathbb{R}_+} \\ \Lambda_0^1 \gamma_1 \Lambda^{-2}_{-,\mathbb{R}_+} \end{pmatrix} : H^s(\mathbb{R}^n_+) \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} .$$

$A\Lambda^{-2}_{-,\mathbb{R}_+}$ is of order zero and $\Lambda_0^1 \gamma_1 \Lambda^{-2}_{-,\mathbb{R}_+}$ is of order $\frac{1}{2}$ and class 0. Then $\mathcal{A}_1^N$ is continuous for all $-\tau < s < \tau$ with $s > -\frac{1}{2}$. By Theorem 2.3.4 $\mathcal{A}_1^N$ has a parametrix $B_1^{0,N}$ of order zero and class 0,

$$\mathcal{B}_1^{0,N} = \begin{pmatrix} R_1^{0,N} & K_1^{0,N} \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to H^s(\mathbb{R}^n_+).$$

Here $R_1^{0,N}$ is of order zero, and $K_1^{0,N}$ is a Poisson operator of order $\frac{1}{2}$, having symbol-kernel in $C^\tau S^{-\frac{1}{2}}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, S(\overline{\mathbb{R}_+}))$. Since $K_1^{0,N}$ has coefficients in C^τ, we have an extra restriction $|s - \frac{1}{2}| < \tau$ on s. Therefore $\mathcal{B}_1^{0,N}$ is continuous for $-\tau + \frac{1}{2} < s < \tau$ and $s > -\frac{1}{2}$. The remainder $\mathcal{R}_1^N = \mathcal{A}_1 \mathcal{B}_1^{0,N} - I$ satisfies

$$\mathcal{R}_1^N = \begin{pmatrix} A\Lambda^{-2}_{-,\mathbb{R}_+} \\ \Lambda_0^1 \gamma_0 \Lambda^{-2}_{-,\mathbb{R}_+} \end{pmatrix} \begin{pmatrix} R_1^{0,N} & K_1^{0,N} \end{pmatrix} - I = \begin{matrix} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s-\theta-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix}$$

when $0 < \theta < \tau$, $-\tau + \frac{1}{2} + \theta < s < \tau - [\theta]$ and $s > -\frac{1}{2} + \theta$.
We obtain

$$\mathcal{A}_1^N \mathcal{B}_1^{0,N} = I + \mathcal{R}_1^N \text{ or } \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^1 \end{pmatrix} \mathcal{A}^N \Lambda^{-2}_{-,\mathbb{R}_+} \mathcal{B}_1^{0,N} = I + \mathcal{R}_1^N.$$

Therefore

$$\mathcal{A}^N \Lambda^{-2}_{-,\mathbb{R}_+} \mathcal{B}_1^{0,N} \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^1 \end{pmatrix} = I + \mathcal{R}'^N, \text{ with } \mathcal{R}'^N = \begin{pmatrix} \mathcal{R}'^N_1 \\ \mathcal{R}'^N_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{-2} \end{pmatrix} \mathcal{R}_1 \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix}.$$

Step 2. Set

$$\mathcal{B}_0^N = \Lambda^{-2}_{-,\mathbb{R}_+} \mathcal{B}_1^{0,N} \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^1 \end{pmatrix} = \begin{pmatrix} \Lambda^{-2}_{-,\mathbb{R}_+} R_1^{0,N} & \Lambda^{-2}_{-,\mathbb{R}^N_+} K_1^{0,N} \Lambda_0^1 \end{pmatrix} = \begin{pmatrix} R_0^N & K_0^N \end{pmatrix}.$$

It is a parametrix of $\mathcal{A}$, and

$$\mathcal{A}^N\mathcal{B}^N = I + \mathcal{R}'^N.$$

$$\mathcal{B}_0^N : \begin{array}{c} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{1}{2}}(\mathbb{R}^{n-1}) \end{array} \to H^{s+2}(\mathbb{R}^n_+),$$

is continuous when

$$-\tau + \frac{1}{2} < s < \tau \text{ and } s > -\frac{1}{2}. \tag{3.7}$$

$$\mathcal{R}'^N = \begin{pmatrix} \mathcal{R}'_1 \\ \mathcal{R}'_2 \end{pmatrix} : \begin{array}{c} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{1}{2}-\theta}(\mathbb{R}^{n-1}) \end{array} \to \begin{array}{c} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{1}{2}}(\mathbb{R}^{n-1}), \end{array}$$

is continuous when

$$0 < \theta < \tau\ , \ -\tau + \frac{1}{2} + \theta < s < \tau - [\theta] \text{ and } s > -\frac{1}{2} + \theta. \tag{3.8}$$

The inequality (3.2) implies that the principal symbol of A takes its values in $\overline{\mathbb{R}_-}$. Now we replace A by $A-\lambda_0$ for a large $|\lambda_0|$ and the parameter $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$ by $\langle\xi\rangle = (1 + |\lambda_0| + |\xi|^2)^{\frac{1}{2}}$ in the order reducing operators. We first prove the theorem for some large λ_0 and then extend it to all $\lambda \in \rho(A^N_{\gamma_1})$.
The parametrix and remainder map as follows:

$$\mathcal{B}_0^N(\lambda_0) = \begin{pmatrix} R_0^N(\lambda_0) & K_0^N(\lambda_0) \end{pmatrix} : \begin{array}{c} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{1}{2}}(\mathbb{R}^{n-1}) \end{array} \to H^{s+2}(\mathbb{R}^n_+), \tag{3.9}$$

$$\mathcal{R}^N(\lambda_0) : \begin{array}{c} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{1}{2}-\theta}(\mathbb{R}^{n-1}) \end{array} \to \begin{array}{c} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac{1}{2}}(\mathbb{R}^{n-1}) \end{array} \tag{3.10}$$

for s and θ as in (3.7) and (3.8).

Consider $\lambda_0 = r^2 e^{i\eta}$, $\eta \in (-\pi, \pi)$ on a ray outside $\overline{\mathbb{R}_-}$. We introduce an extra variable $t \in S^1$ (S^1 is the unit circle), and replace r by $D_t = -i\partial_t$, now consider

$$\tilde{A} = A - e^{i\eta}D_t^2 \text{ on } \mathbb{R}^n_+ \times S^1.$$

Since $a(x,\xi) - \lambda_0 \neq 0$, $\tilde{A}$ is elliptic on $\mathbb{R}^+ \times S^1$ and $\tilde{\mathcal{A}}^N = \begin{pmatrix} \tilde{A} \\ \gamma_1 \end{pmatrix}$ has a parametrix $\tilde{\mathcal{B}}_0^N$ and the remainder $\tilde{\mathcal{R}}^N = \tilde{\mathcal{A}}^N\tilde{\mathcal{B}}_0^N - I$ as in (3.9), (3.10) with $\mathbb{R}^n_+$, $\mathbb{R}^{n-1}$ replaced by $\mathbb{R}^n_+ \times S^1$, $\mathbb{R}^{n-1} \times S^1$. For functions w of the form $w(x,t) = u(x)e^{ir_0t}$, $u \in S(\mathbb{R}^n_+)$ and $r_0 \in 2\pi\mathbb{Z}$, we have

$$\tilde{\mathcal{A}}^N = \begin{pmatrix} (A - e^{i\eta}r_0^2)w \\ \gamma_1 w \end{pmatrix} = \begin{pmatrix} (A - \lambda_0)w \\ \gamma_1 w \end{pmatrix},$$

$$\|w\|_{H^s(\mathbb{R}^n_+\times S^1)} \simeq \|(1+|\xi|^2+r_0^2)^{\frac{s}{2}}\hat{u}(\xi)\|_{L_2} = \|u\|_{H^{s,r_0}}.$$

When $s' < s$, then

$$\begin{aligned}\|w\|_{H^{s'}(\mathbb{R}^n_+\times S^1)} &\simeq \|(1+|\xi|^2+r_0^2)^{\frac{s'}{2}}\hat{u}(\xi)\|_{L_2}\\ &\leq \langle r_0\rangle^{s'-s}\|(1+|\xi|^2+r_0^2)^{\frac{s}{2}}\hat{u}(\xi)\|_{L_2} \simeq \langle r_0\rangle^{s'-s}\|w\|_{H^s(R^n_+\times S^1)}.\end{aligned}$$

Since the remainder $\tilde{\mathcal{R}}^N$ acts like $\mathcal{R}^N(\lambda_0)$, we find that

$$\begin{aligned}\|\mathcal{R}^N(\lambda_0)\{u_1,u_2\}\|_{H^{s,r_0}(\mathbb{R}^n_+)\times H^{s+\frac{1}{2},r_0}(\mathbb{R}^{n-1})} &\leq c_s\|\{u_1,u_2\}\|_{H^{s-\theta,r_0}(\mathbb{R}^n_+)\times H^{s+\frac{1}{2}-\theta,r_0}(\mathbb{R}^{n-1})} \quad (3.11)\\ &\leq c_s'\langle r_0\rangle^{-\theta}\|\{u_1,u_2\}\|_{H^{s,r_0}(\mathbb{R}^n_+)\times H^{s+\frac{1}{2},r_0}(\mathbb{R}^{n-1})}\end{aligned}$$

for s as in (3.8). Now we want to extend it to arbitrary λ as follows: Write $\lambda = r^2e^{i\eta} = (r_0+r')^2e^{i\eta}$ with $r' \in [0, 2\pi)$. Since

$$(1+|\xi|^2+r_0^2)^{\frac{t}{2}} \simeq (1+|\xi|^2+(r_0+r')^2)^{\frac{t}{2}}, \qquad (3.12)$$

$$A-\lambda = A-\lambda_0+(\lambda_0-\lambda),$$

$$|\lambda_0-\lambda| = |r_0^2-r^2| = |2r_0r'+r'^2| \leq c\langle r_0\rangle, \qquad (3.13)$$

$$\|\mathcal{B}_0^N(\lambda_0)\{u_1,u_2\}\|_{H^{s+2,r}(\mathbb{R}_+)} \leq c_s\|\{u_1,u_2\}\|_{H^{s,r}(\mathbb{R}_+)\times H^{s+\frac{1}{2},r}(\mathbb{R}^{n-1})}, \qquad (3.14)$$

$$\begin{aligned}\|\mathcal{B}_0^N(\lambda_0)\{u_1,u_2\}\|_{H^{s+2,r}(\mathbb{R}_+)} &\simeq \|(1+|\xi|^2+|r_0|^2)^{\frac{s+2}{2}}\mathcal{B}_0^N(\lambda_0)\{u_1,u_2\}\|_{L_2}\\ &\geq \langle r_0\rangle\|(1+|\xi|^2+|r_0|)^{\frac{s+2}{2}}\mathcal{B}_0^N(\lambda_0)\{u_1,u_2\}\|_{L_2}\\ &\geq \langle r_0\rangle\|(1+|\xi|^2+|r_0|)^{\frac{s}{2}}\mathcal{B}_0^N(\lambda_0)\{u_1,u_2\}\|_{L_2}\\ &= \langle r_0\rangle\|\mathcal{B}_0^N(\lambda_0)\{u_1,u_2\}\|_{H^{s,r}(\mathbb{R}_+)}, \qquad (3.15)\end{aligned}$$

we have

$$\mathcal{R}^N(\lambda) = \mathcal{A}^N(\lambda)\mathcal{B}_0^N(\lambda_0) - I = \mathcal{A}^N(\lambda_0)\mathcal{B}_0^N(\lambda_0) - I + \begin{pmatrix}\lambda_0-\lambda\\ 0\end{pmatrix}\mathcal{B}_0^N(\lambda_0). \qquad (3.16)$$

Therefore

$$\begin{aligned}\|\mathcal{R}^N(\lambda)\{u_1,u_2\}\|_{H^{s,r}(\mathbb{R}^n_+)\times H^{s+\frac{1}{2},r}(\mathbb{R}^{n-1})} &\overbrace{\simeq}^{(3.12)} \|\mathcal{R}^N(\lambda)\{u_1,u_2\}\|_{H^{s,r_0}(\mathbb{R}^n_+)\times H^{s+\frac{1}{2},r_0}(\mathbb{R}^{n-1})}\\ &\overbrace{\leq}^{(3.16)} \|\mathcal{R}^N(\lambda_0)\{u_1,u_2\}\|_{H^{s,r_0}(\mathbb{R}^n_+)\times H^{s+\frac{1}{2},r_0}(\mathbb{R}^{n-1})}\\ &\quad +|\lambda_0-\lambda|\ \|\mathcal{B}_0^N(\lambda_0)\{u_1,u_2\}\|_{H^{s,r}(\mathbb{R}_+)}\\ &\overbrace{\leq}^{(3.11),(3.15),(3.14)} d_s\|\{u_1,u_2\}\|_{H^{s,r_0}(\mathbb{R}^n_+)\times H^{s+\frac{3}{2},r_0}(\mathbb{R}^{n-1})}\end{aligned}$$

If we define $B_0^N(\lambda) = B_0^N(\lambda_0)$, then (3.11) holds for general λ.
Fix s. Consider $\lambda = r^2e^{i\eta}$ with $r_0 \geq r_1$ for large r_1, where for each s, $c_s'\langle r_0\rangle^{-\theta} \leq$

$\frac{1}{2}$. Then $\|\mathcal{R}^N(\lambda)\|_{H^{s,r}\times H^{s+\frac12,r}} < 1$ and $I+\mathcal{R}^N(\lambda)$ has the inverse $I+\mathcal{R}''^N(\lambda) = I+\sum_{k\geq 1}(-\mathcal{R}^N(\lambda))^k$ (converging in the operator norm for operators on $H^{s,r}(\mathbb{R}^n_+)\times H^{s+\frac12,r}(\mathbb{R}^{n-1})$). Therefore by definition of $\mathcal{B}_0^N(\lambda)$,

$$\mathcal{A}^N(\lambda)\mathcal{B}_0^N(\lambda)(I+\mathcal{R}''^N(\lambda)) = I.$$

This gives the right inverse

$$\begin{aligned}\mathcal{B}^N(\lambda) &= \mathcal{B}_0^N(\lambda)+\mathcal{B}_0^N(\lambda)\mathcal{R}''^N(\lambda)\\ &= \begin{pmatrix}R_0^N(\lambda) & K_0^N(\lambda)\end{pmatrix}+\begin{pmatrix}R_0^N(\lambda) & K_0^N(\lambda)\end{pmatrix}\mathcal{R}''^N(\lambda) = \begin{pmatrix}R^N(\lambda) & K^N(\lambda)\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{B}_0^N(\lambda)\mathcal{R}''^N(\lambda)\{f,g\}\|_{H^{s+2,r}(\mathbb{R}^n_+)} &\leq c_s\|\{f,g\}\|_{H^{s-\theta,r}(\mathbb{R}^n_+)\times H^{s+\frac12-\theta,r}(\mathbb{R}^{n-1})}\\ &\leq c_s'\langle r\rangle^{-\theta}\|\{f,g\}\|_{H^{s,r}(\mathbb{R}^n_+)\times H^{s+\frac12,r}(\mathbb{R}^{n-1})}.\end{aligned}$$

Since

$$\mathcal{A}^N(\lambda)\mathcal{B}^N(\lambda) = \begin{pmatrix}(A-\lambda)R^N(\lambda) & (A-\lambda)K^N(\lambda)\\ \gamma_1 R^N(\lambda) & \gamma_1 K^N(\lambda)\end{pmatrix} = \begin{pmatrix}I & 0\\ 0 & I\end{pmatrix},$$

$R^N(\lambda)$ solves

$$(A-\lambda)u = f\ ,\ \gamma_1 u = 0, \tag{3.17}$$

and $K^N(\lambda)$ solves

$$(A-\lambda)u = 0\ ,\ \gamma_1 u = \varphi. \tag{3.18}$$

For such large λ, $R^N(\lambda)$ coincides with the resolvent $(A^N_{\gamma_1}-\lambda)^{-1}$ of $A^N_{\gamma_1}$. The operator $K^N(\lambda)$ is the Poisson operator, which solves (3.18). Since $\lambda\in\rho(A^N_{\gamma_1})$, it is denoted by $K^\lambda_{\gamma_1}$. For each $\lambda = r^2e^{i\eta}$, $r_1\leq r$,

$$(A^N_{\gamma_1}-\lambda)^{-1}: H^s(\mathbb{R}^n_+)\to H^{s+2}(\mathbb{R}^n_+)\ ,\ K^\lambda_{\gamma_1}: H^{s+\frac12}(\mathbb{R}^{n-1})\to H^{s+2}(\mathbb{R}^n_+)$$

for s satisfying (3.7).

From above

$$R^N(\lambda) = R_0^N(\lambda)+R_0^N(\lambda)\mathcal{R}''^N(\lambda)\ ,\ K^N(\lambda) = K^\lambda_{\gamma_1} = K_0^N(\lambda)+K_0^N(\lambda)\mathcal{R}''^N(\lambda) \tag{3.19}$$

where

$$\|\ R_0^N(\lambda)\ \|_{\mathcal{L}(H^{s,r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))}\quad ,\quad \|\ K_0^N(\lambda)\ \|_{\mathcal{L}(H^{s+\frac12,r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))} \tag{3.20}$$

are $O(1)$,

$$\|\ R_0^N(\lambda)\mathcal{R}''^N(\lambda)\ \|_{\mathcal{L}(H^{s-\theta,r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))}\quad ,\quad \|\ K_0^N(\lambda)\mathcal{R}''^N(\lambda)\ \|_{\mathcal{L}(H^{s+\frac12-\theta,r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))} \tag{3.21}$$

are $O(1)$, and

$$\| R_0^N(\lambda)\mathcal{R}''^N(\lambda) \|_{\mathcal{L}(H^{s,r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))} \quad , \quad \| K^{N_0}(\lambda)\mathcal{R}''^N(\lambda) \|_{\mathcal{L}(H^{s+\frac{1}{2},r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))} \tag{3.22}$$

are $O(\langle\lambda\rangle^{-\frac{\theta}{2}})$.

For λ on the rays $\lambda = r^2 e^{i\eta}$ as in the statement of the theorem and s as in (3.7) and (3.8). In view of [8, A.26] we have

$$\begin{aligned}
\|R^N(\lambda)u_1\|_{H^{s+2,r}} &\cong \Big(\langle\lambda\rangle^{s+2}\|R^N(\lambda)u_1\|_0^2 + \|R^N(\lambda)u_1\|^2_{H^{s+2}}\Big)^{\frac{1}{2}} \\
&\cong C'\langle\lambda\rangle^{\frac{s}{2}+1}\|R^N(\lambda)u_1\|_0^2 + C'\|R^N(\lambda)u_1\|^2_{H^{s+2}} \\
&\leq \|u_1\|_{H^{s,r}} \cong C_s\Big(\langle\lambda\rangle^{s}\|u_1\|_0^2 + \|u_1\|^2_{H^s}\Big)^{\frac{1}{2}} \\
&\leq C'_s\langle\lambda\rangle^{\frac{s}{2}}\|u_1\|_{H^s} \qquad (3.23) \\
\|K^N(\lambda)u_2\|_{H^{s+2,r}} &\cong \Big(\langle\lambda\rangle^{s+2}\|K^N(\lambda)u_2\|_0^2 + \|K^N(\lambda)u_2\|^2_{H^{s+2}}\Big)^{\frac{1}{2}} \\
&\cong C'\langle\lambda\rangle^{\frac{s}{2}+1}\|K^N(\lambda)u_2\|_0^2 + C'\|K^N(\lambda)u_2\|^2_{H^{s+2}} \\
&\leq \|u_2\|_{H^{s+\frac{1}{2},r}} \cong C_s\Big(\langle\lambda\rangle^{s+\frac{1}{2}}\|u_2\|_0^2 + \|u_2\|^2_{s+\frac{1}{2}}\Big)^{\frac{1}{2}}. \qquad (3.24)
\end{aligned}$$

Note that $\|R^N(\lambda)\|_{\mathcal{L}(L^2(\mathbb{R}^n_+))}$ is $O(\langle\lambda\rangle^{-1})$ on the ray. □

From the result of Theorem 3.2.2 we easily obtain the structure of the resolvent for compact manifolds with boundary. We state it for the sake of completeness.

Let $\overline{D}$ be a smooth compact manifold with boundary. We have the operator A as in (3.1). Consider the situation, where $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1}$ are replaced by $\overline{D}$ and ∂D.

Theorem 3.2.3. *Let A be as* (3.1) *satisfying conditions (1), (2) and (3). Then for $\lambda \in \rho(A^N_{\gamma_1})$, $-\tau + \frac{1}{2} < s < \tau$ and $s > -\frac{1}{2}$, the operator*

$$\begin{pmatrix} A-\lambda \\ \gamma_1 \end{pmatrix} : H^{s+2}(\overline{D}) \to \begin{matrix} H^s(\overline{D}) \\ \times \\ H^{s+\frac{1}{2}}(\partial D) \end{matrix} \quad ;$$

has an inverse

$$\Big(R^N(\lambda) \quad K^N(\lambda)\Big) = \Big((A^N_{\gamma_1}-\lambda)^{-1} \quad K^\lambda_{\gamma_1}\Big) : \begin{matrix} H^s(\overline{D}) \\ \times \\ H^{s+\frac{1}{2}}(\partial D) \end{matrix} \to H^{s+2}(\overline{D}).$$

On the rays $\lambda = re^{i\eta}$ with $\eta \in (\frac{\pi}{2}-\epsilon, \frac{3\pi}{2}+\epsilon)$ for $0 < \epsilon < \frac{\pi}{2}$ (Outside the range of the principal symbol), the inverse exists for $|\lambda|$ sufficiently large. The operators $R^N(\lambda)$ and $K^N(\lambda)$ have the structure in (3.19) *and satisfy estimates* (3.20)-(3.24) .

Proof. We can show it as in the proof of Theorem 3.1.3. □

Chapter 4

Regularity of a Degenerate Boundary Value Problem

In this section we consider the degenerate boundary value problem for second order uniformly strongly elliptic differential operators with non-smooth coefficients which includes as particular cases the Dirichlet and Neumann problems. In the case of smooth coefficients, it has been proved by Taira in [18] in the framework of Sobolev spaces and in [20] in the framework of Hölder spaces.
We generalize this result to the case of non-smooth coefficients for $\tau \geq 4$ in the framework of Sobolev spaces.
A key to finding a unique solution for the degenerate boundary value problem is first the idea that we can reduce the study of this problem to that of a pseudodifferential operator $T(\lambda)$ on the boundary. Next we fix $\tilde{\lambda}$ and show that the symbol of $T(\tilde{\lambda})$ - we can actually focus on $t_1 + t_0$, the first two terms in the asymptotic expansion - satisfies the hypoellipticity assumption of Definition (2.5.1). We find a parametrix $Op(b_0(\tilde{\mu}))$ to $T(\tilde{\lambda}) - \tilde{\mu}$ with $b_0(\tilde{\mu}) \in C^{\tau-1}S^0_{1,\frac{1}{2}}$. In particular, this yields an a priori estimate for $T(\tilde{\mu})$, (4.31). Then we consider the realization A_L of A with boundary condition L and prove a priori estimates for $A_L - \lambda$ (Theorem 4.3.1), using the a priori estimates for $T(\tilde{\lambda})$. For $|\lambda|$ sufficiently large, they imply that $A_L - \lambda$ is injective. Since

$$N\Big((A-\lambda, L)\Big) = \Big\{u \in H_L^{s+2}; (A-\lambda, L)u = 0\Big\} = N(A_L - \lambda),$$

we conclude that $T(\lambda)$ is injective and hence invertible.

Let $\overline{D} = D \cup \partial D$ be an n-dimensional, compact C^∞ manifold with boundary. Let

$$A(x, D) = \sum_{i,j=1}^{n} a^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b^i(x)\partial_{x_i} + c(x)$$

be a second-order elliptic differential operator with real coefficients. Fix $\tau \geq 4$. We assume

A.1) $a^{ij} \in C^{\tau}(\overline{D})$, $a^{ij} = a^{ji}$ and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, x \in \overline{D}, \xi \in \mathbb{R}^n; \tag{4.1}$$

A.2) $b^i \in C^{\tau}(\overline{D})$

A.3) $c \in C^{\tau}(\overline{D})$ and $c \leq 0$ on $\overline{D}$ but $c \neq 0$ in D.

We consider the following boundary value problem: Given functions f and φ defined in D and ∂D, respectively, find a function u in D such that

$$\begin{cases} (A - \lambda)u = f & in\ D \\ Lu = \mu_1(x')\frac{\partial u}{\partial n}(x') + \mu_2(x')u(x') = \varphi & on\ \partial D \end{cases} \tag{4.2}$$

Here:
1) λ is a complex number in the sector

$$\left\{\lambda = r^2 e^{i\eta}, -\pi + \epsilon < \eta < \pi - \epsilon\right\} \tag{4.3}$$

for some $0 < \epsilon < \pi$.
2) $\mu_1 \in C^{\tau}(\partial D)$ and $\mu_1 \geq 0$ on ∂D
3) $\mu_2 \in C^{\tau}(\partial D)$ and $\mu_2 \geq 0$ on ∂D
4) $n = (n_1, \cdots, n_n)$ is the unit exterior normal to ∂D
5) $\frac{\partial}{\partial n}$ the normal derivative.

Our fundamental hypothesis is the following :

$$\mu_1(x') + \mu_2(x') > 0,\ x' \in \partial D\ . \tag{4.4}$$

We consider $(A - \lambda, L)$ as an unbounded operator in H^{s+2} with domain $H_L^{s+2} = \left\{u \in H^{s+2}; Lu \in H^{s+\frac{3}{2}}\right\}$. Our main result here is:

Theorem 4.0.4. *Under the above assumptions on A and L, the mapping*

$$\begin{pmatrix} A - \lambda \\ L \end{pmatrix} : H_L^{s+2}(\overline{D}) \to H^s(\overline{D}) \times H^{s+3/2}(\partial D) \tag{4.5}$$

is a topological isomorphism for all

$$0 \leq s < \tau - 3,$$

for all λ in (4.3), *$|\lambda|$ sufficiently large.*

It is easy to see that $(A - \lambda, L)$ is closed: Let $u_n \in H_L^{s+2}$ with $u_n \to u$ in H^{s+2}, $(A - \lambda)u_n \to v$ in H^s and $Lu_n \to g$ in $H^{s+\frac{3}{2}}$. Since $Lu_n \to Lu$ in $H^{s+\frac{1}{2}}$, then $Lu = g \in H^{s+\frac{3}{2}}$ and $u \in H_L^{s+2}$. Further the operator $(A - \lambda, L)$ is densely defined, since the domain $H_L^{s+2}(\overline{D})$ contains $C^{\infty}(\overline{D})$ and so it is dense in $H^{s+2}(\overline{D})$.

The proof will be proven in the remainder of this section.

We reduce the analysis of (4.5) to the analysis of pseudodifferential operator on the boundary. According to the Sections 3.1 and 3.2, there exist $\begin{pmatrix} R^D(\lambda) & K^D(\lambda) \end{pmatrix}$ and $\begin{pmatrix} R^N(\lambda) & K^N(\lambda) \end{pmatrix}$ such that

$$\begin{pmatrix} A-\lambda \\ \gamma_0 \end{pmatrix}\begin{pmatrix} R^D(\lambda) & K^D(\lambda) \end{pmatrix} = I\,, \quad \begin{pmatrix} A-\lambda \\ \gamma_1 \end{pmatrix}\begin{pmatrix} R^N(\lambda) & K^N(\lambda) \end{pmatrix} = I$$

for all λ in (4.3). Let $L = (\mu_1 \frac{\partial}{\partial n} + \mu_2)|_{\partial D}$. Then

$$\begin{pmatrix} A-\lambda \\ L \end{pmatrix}\begin{pmatrix} R^N(\lambda) & K^D\lambda) \end{pmatrix} = \begin{pmatrix} I & 0 \\ LR^N(\lambda) & LK^D(\lambda) \end{pmatrix} : \begin{matrix} H^s(\overline{D}) \\ \times \\ D(T(\lambda)) \end{matrix} \to \begin{matrix} H^s(\overline{D}) \\ \times \\ H^{s+\frac{3}{2}}(\partial D) \end{matrix} \tag{4.6}$$

with the operator $T(\lambda) = LK^D(\lambda)$ considered as an unbounded operator in $H^{s+\frac{3}{2}}(\partial D)$ with domain $D(T(\lambda)) = \left\{\psi \in H^{s+\frac{3}{2}}(\partial D); T(\lambda)\psi \in H^{s+\frac{3}{2}}(\partial D)\right\}$. The idea to use $R^N(\lambda)$ in (4.6) is that, since it solves (3.17), then

$$LR^N(\lambda) = \mu_1(x')\gamma_1 R^N(\lambda) + \mu_2(x')\gamma_0 R^N(\lambda) = \mu_2(x')\gamma_0 R^N(\lambda)$$

is an operator in $H^{s+\frac{3}{2}}(\partial D)$. The operator on the right hand side (4.6) has an inverse if and only if $T(\lambda) = LK^D(\lambda)$ is invertible; in that case the inverse is given by

$$\begin{pmatrix} I & 0 \\ -(LK^D(\lambda))^{-1}(LR^N(\lambda)) & (LK^D(\lambda))^{-1} \end{pmatrix}.$$

Note that $T(\lambda) = LK^D(\lambda) = \mu_2 I + \mu_1\Lambda(\lambda)$ with $\Lambda(\lambda) = \gamma_1 K^D(\lambda)$ the Dirichlet-to-Neumann operator. Thus one can reduce the study of problem (4.2) to that pseudodifferential operator $T(\lambda)$ on the boundary. More precisely, we have the following:

Lemma 4.0.5. *The boundary problem* (4.5) *is invertible if and only if* (4.6) *is invertible and* (4.6) *is invertible if and only if the operator* $T(\lambda) : D(T(\lambda)) \subset H^{s+\frac{3}{2}}(\partial D) \to H^{s+\frac{3}{2}}(\partial D)$ *is invertible.*

In order to study the operator $T(\lambda)$ we need some notation. In a neighborhood of boundary we can write the operator $A = A(x, D)$ in the form

$$A(x,D) - \lambda = \sum_{k\le 2} A_k D^k_{x_n} = A_2(x)D^2_{x_n} + A_1(x, D_{x'})D_{x_n} + A_0(x, D_{x'}) - \lambda, \tag{4.7}$$

$$x = (x', x_n)$$

where

$$\begin{cases} A_0(x, D_{x'}) - \lambda = -\sum\limits_{i,j<n} a^{ij}(x)D_{x_i}D_{x_j} + i\sum\limits_{i=1}^{n-1} b^i(x)D_{x_i} + c(x) - \lambda \\ A_1(x, D_{x'}) = -\sum\limits_{i=1}^{n-1} a^{in}(x)D_{x_i} + ib^n(x) \\ A_2(x) = -a_{nn} \end{cases}$$

are differential operators of order 2, 1 and zero respectively, and their symbols are

$$\begin{cases} \sigma(A_0-\lambda) = -\sum\limits_{i,j<n} a^{ij}(x)\xi_i\xi_j + i\sum\limits_{i=1}^{n-1} b^i(x)\xi_i + c(x) - \lambda \\ \sigma(A_1) = -\sum\limits_{i=1}^{n-1} a^{in}(x)\xi_i + ib^n(x) \\ \sigma(A_2) = -a_{nn}. \end{cases}$$

We denote by $a_1(x,\xi')$ and $a_0(x,\xi')$ the principal symbols of $A_1(x,D_{x'})$ and $A_0(x,D_{x'})$, respectively. We can conclude the following from (4.1):

b.1) $A_2(x) < 0$

The principal symbol of A has two roots with respect to ξ_n and has the following decomposition:

$$\begin{aligned} a_2 - \lambda = -\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j - \lambda &= A_2(x)\xi_n^2 + a_1(x,\xi')\xi_n + a_0(x,\xi') - \lambda \\ &= A_2(x)(\kappa_1(x,\xi',\lambda) - \xi_n)(\kappa_2(x,\xi',\lambda) - \xi_n) \quad (4.8) \end{aligned}$$

where:

$$\begin{aligned} \kappa_2(x,\xi',\lambda) &= -\frac{a_1(x,\xi')}{2A_2(x)} + i\frac{(4A_2(x)(a_0(x,\xi')-\lambda) - a_1(x,\xi')^2)^{\frac{1}{2}}}{2A_2(x)} \\ &= q_1(x,\xi') + ip_1(x,\xi',\lambda) \text{ in } \mathbb{C}_- \qquad (4.9) \end{aligned}$$

$$\begin{aligned} \kappa_1(x,\xi',\lambda) &= -\frac{a_1(x,\xi')}{2A_2(x)} - i\frac{(4A_2(x)(a_0(x,\xi')-\lambda) - a_1(x,\xi')^2)^{\frac{1}{2}}}{2A_2(x)} \\ &= q_1(x,\xi') - ip_1(x,\xi',\lambda) \text{ in } \mathbb{C}_+ \qquad (4.10) \end{aligned}$$

since $A_2 \in C^\tau(\overline{D})$, $a_1(x,\xi') \in C^\tau S^1_{1,0}(\overline{D}\times\mathbb{R}^{n-1})$ and $a_0(x,\xi') \in C^\tau S^2_{1,0}(\overline{D}\times\mathbb{R}^{n-1})$, it follows that:
1) $q_1(x,\xi') \in C^\tau S^1_{1,0}(\overline{D}\times\mathbb{R}^{n-1})$
2) $4A_2(x)(a_0(x,\xi')-\lambda) - a_1(x,\xi')^2 \in C^\tau S^2_{1,0}(\overline{D}\times\mathbb{R}^{n-1})$.
Hence $\kappa_1(x,\xi',\lambda)$ and $\kappa_2(x,\xi',\lambda)$ by Lemma 2.1.10 are in $C^\tau S^1_{1,0}(\overline{D}\times\mathbb{R}^{n-1})$.

4.1 The Calderón Projector

Next we write

$$\begin{pmatrix} A-\lambda \\ \gamma_0 \end{pmatrix}^{-1} = \begin{pmatrix} R^D(\lambda) & K^D(\lambda) \end{pmatrix} = \begin{pmatrix} Q_\lambda + G_\lambda & K^D(\lambda), \end{pmatrix} \qquad (4.11)$$

where Q_λ is an inverse of $A-\lambda$ and G_λ is a singular Green operator of order -2 and class 0. The operators

$$R^D(\lambda) : H^s(\overline{D}) \to H^{s+2}(\overline{D}),\ K^D(\lambda) : H^{s+\frac{3}{2}}(\partial D) \to H^{s+2}(\partial D)$$

are bounded for every s satisfying

$$-\tau+\frac{1}{2}<s<\tau \text{ and } s>-\frac{3}{2}.$$

The operator A satisfies the following Green's formula for $u,v\in C_0^\infty(\overline{D})$:

$$\langle Au,v\rangle_D-\langle u,A^*v\rangle_D=\langle\mathfrak{A}\rho u,\rho v\rangle_{\partial D},$$

$$\mathfrak{A}=\begin{pmatrix} iA_1(x',D_{x'})+iD_nA_2(x')-i\sum\limits_{i=1}^{n-1}D_ia^{in}(x) & iA_2(x')\\ iA_2(x') & 0\end{pmatrix}.$$

The adjoint of the two sided trace operator

$$\rho=\begin{pmatrix}\gamma_0\\ \gamma_1=\gamma_0D_n\end{pmatrix}:H^{s+2}(D)\to H^{s+\frac{3}{2}}(\partial D)\times H^{s+\frac{1}{2}}(\partial D)\ ,\ s>-\frac{1}{2};$$

is the operator

$$\rho^*=\begin{pmatrix}\gamma_0^* & \gamma_1^*=D_n^*\gamma_0^*\end{pmatrix}:H^{-s-\frac{3}{2}}(\partial D)\times H^{-s-\frac{1}{2}}(\partial D)\to H^{-s-2}(D)\ ,\ s>-\frac{1}{2}$$

here $\gamma_0^*\varphi=\varphi(x')\otimes\delta(x_n)$ is a distribution supported in ∂D. Define the nullspace

$$N(A-\lambda)=\left\{u\in H^{s+2}(D);(A-\lambda)u=0 \text{ on } D\right\}.$$

The operator

$$\tilde{K}_\lambda=Q_\lambda\rho^*\mathfrak{A}:H^{s+\frac{3}{2}}(\partial D)\times H^{s+\frac{1}{2}}(\partial D)\to H^{s+2}(D)$$

maps into $N(A-\lambda)$ since

$$(A-\lambda)\tilde{K}_\lambda\varphi=(A-\lambda)Q_\lambda\rho^*\mathfrak{A}\varphi=\rho^*\mathfrak{A}\varphi$$

is supported in ∂D. Hence

$$\tilde{K}_{\lambda,+}=-r^+Q_\lambda\rho^*\mathfrak{A}:H^{s+\frac{3}{2}}(\partial D)\times H^{s+\frac{1}{2}}(\partial D)\to H^{s+2}(D)$$

acts as a left inverse of ρ. Cf. [8, p. 42]
The operator $C_+=\rho\tilde{K}_{\lambda,+}$ is a pseudodifferential projection in $H^{s+\frac{3}{2}}(\partial D)\times H^{s+\frac{1}{2}}(\partial D)$, the Calderón projector.

Now we study the symbols of Q_λ, which is the inverse of $A-\lambda$. Write $\sigma(Q_\lambda)\sim\sum r_{-2-j}$ and $\sigma(A-\lambda)\sim\sum a_{2-k}$. Then

$$\sigma(A-\lambda)\sharp\sigma(Q_\lambda)=I,$$

$\sharp$ is the Leibniz product. We have

$$(a_2-\lambda)r_{-2}=1 \tag{4.12}$$

and

$$r_{-2-l}=-r_{-2}\sum\frac{1}{\alpha!}D_\xi^\alpha a_{2-k}\partial_x^\alpha r_{-2-j}$$

with the sum taken for $j < l$, $j + |\alpha| + k = l$ and $l = 1, 2, .. < \tau$. Here we will confine ourselves to $l = 0, 1$ and compute only the first two components of the symbol of Q_λ. They are

$$\begin{aligned} r_{-2}(x,\xi,\lambda) &= \frac{1}{a_2-\lambda} \in C^\tau S^{-2}_{1,0} \\ r_{-3}(x,\xi,\lambda) &= -r_{-2}\sum_{j=1}^{n} D_{\xi_j} a_2 \partial_{x_j} r_{-2} - r_{-2} a_1 r_{-2} \in C^{\tau-1} S^{-3}_{1,0}. \end{aligned}$$

Then

$$r_{-2}(x, D_x, \lambda) : H^{s-2}(\overline{D}) \to H^s(\overline{D})$$

is continuous if $-\tau < s < \tau$ and

$$r_{-3}(x, D_x, \lambda) : H^{s-3}(\overline{D}) \to H^s(\overline{D})$$

is continuous if $-\tau + 1 < s < \tau - 1$. Set

$$Q_\lambda = Op(r_{-2} + r_{-3}) + \mathcal{R}_1(\lambda) = \underline{Q_\lambda} + \mathcal{R}_1(\lambda).$$

If we insert the precise form of $a_2 - \lambda$ i.e. (4.8) in r_{-2} and r_{-3}, we have

$$r_{-2}(x,\xi,\lambda) = \frac{1}{A_2(x)(\kappa_1-\xi_n)(\kappa_2-\xi_n)} \tag{4.13}$$

$$\begin{aligned} r_{-3}(x,\xi,\lambda) = & -\sum_{j=1}^{n}\Big(\frac{\partial_{x_j}A_2(x) D_{\xi_j}(\kappa_1-\xi_n)}{A_2^2(x)(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)} + \frac{\partial_{x_j}A_2(x) D_{\xi_j}(\kappa_2-\xi_n)}{A_2^2(x)(\kappa_1-\xi_n)(\kappa_2-\xi_n)^2} \\ & + \frac{D_{\xi_j}(\kappa_1-\xi_n)\partial_{x_j}(\kappa_1-\xi_n)}{A_2(x)(\kappa_1-\xi_n)^3(\kappa_2-\xi_n)} + \frac{D_{\xi_j}(\kappa_2-\xi_n)\partial_{x_j}(\kappa_1-\xi_n)}{A_2(x)(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)^2} \\ & + \frac{D_{\xi_j}(\kappa_1-\xi_n)\partial_{x_j}(\kappa_2-\xi_n)}{A_2(x)(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)^2} + \frac{D_{\xi_j}(\kappa_2-\xi_n)\partial_{x_j}(\kappa_2-\xi_n)}{A_2(x)(\kappa_1-\xi_n)(\kappa_2-\xi_n)^3}\Big) \\ & - \frac{a_1(x',\xi')}{A_2^2(x)(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)^2}. \end{aligned} \tag{4.14}$$

Composition of γ_1 with the operator $K^D(\lambda)$ in (4.11) i.e. $\Lambda(\lambda) = \gamma_1 K^D(\lambda)$ is a first order pseudodifferential operator. It is called often Dirichlet-to-Neumann operator and maps $H^{s+\frac{3}{2}}(\partial D)$ to $H^{s+\frac{1}{2}}(\partial D)$.

Now we want to construct the first two terms in the asymptotic expansion of the symbol of $\Lambda(\lambda)$ from the symbols of the Calderón projector. Let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ in $H^{s+\frac{3}{2}}(\partial D) \times H^{s+\frac{1}{2}}(\partial D)$, then

$$\begin{aligned} C_{\lambda,+}f &= \begin{pmatrix} C^+_{\lambda,00} & C^+_{\lambda,01} \\ C^+_{\lambda,10} & C^+_{\lambda,11} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix} Q_\lambda \begin{pmatrix} \gamma_0^* & \gamma_1^* \end{pmatrix} \begin{pmatrix} iA_1(x', D_{x'}) & iA_2(x') \\ iA_2(x') & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_0 Q_\lambda \gamma_0^* iA_1(x', D_{x'}) f_1 + i\gamma_0 Q_{\lambda,+}\gamma_0^* A_2(x') f_2 + \gamma_0 Q_\lambda \gamma_1^* iA_2(x') f_1 \\ \gamma_1 Q_\lambda \gamma_0^* iA_1(x', D_{x'}) f_1 + i\gamma_1 Q_{\lambda,+}\gamma_0^* A_2(x') f_2 + \gamma_1 Q_\lambda \gamma_1^* iA_2(x') f_1 \end{pmatrix} \end{aligned} \tag{4.15}$$

Hence

$$\begin{cases} C^+_{\lambda,00} = \gamma_0 Q_\lambda \gamma_0^* iA_1(x', D_{x'}) + \gamma_0 Q_\lambda \gamma_1^* iA_2(x') \\ C^+_{\lambda,01} = i\gamma_0 Q_\lambda \gamma_0^* A_2(x') \\ C^+_{\lambda,10} = \gamma_1 Q_\lambda \gamma_0^* iA_1(x', D_{x'}) + \gamma_1 Q_\lambda \gamma_1^* iA_2(x') \\ C^+_{\lambda,11} = i\gamma_1 Q_\lambda \gamma_0^* A_2(x'). \end{cases}$$

We insert $\underline{Q_\lambda}$ instead of Q_λ to construct the first two terms in the asymptotic expansion of the symbols of

$$\begin{cases} \underline{C^+_{\lambda,00}} = \gamma_0 \underline{Q_\lambda} \gamma_0^* iA_1(x', D_{x'}) + \gamma_0 \underline{Q_\lambda} \gamma_1^* iA_2(x') \\ \underline{C^+_{\lambda,01}} = i\gamma_0 \underline{Q_\lambda} \gamma_0^* A_2(x') \\ \underline{C^+_{\lambda,10}} = \gamma_1 \underline{Q_\lambda} \gamma_0^* iA_1(x', D_{x'}) + \gamma_1 \underline{Q_\lambda} \gamma_1^* iA_2(x') \\ \underline{C^+_{\lambda,11}} = i\gamma_1 \underline{Q_\lambda} \gamma_0^* A_2(x') \end{cases}$$

from Theorem 2.4.2 and the Residue Theorem as follows:

$$\begin{aligned} \underline{C^+_{\lambda,00}} f_1 &= \gamma_0 \underline{Q_\lambda} \gamma_0^* iA_1(x', D_{x'}) f_1 + \gamma_0 \underline{Q_\lambda} \gamma_1^* iA_2(x') f_1 \\ &= \gamma_0 (r_{-2}(x, D_x, \lambda) + r_{-3}(x, D_x, \lambda))(iA_1(x', D_{x'}) f_1 \otimes \delta(x_n)) \\ &\quad + \gamma_0 (r_{-2}(x, D_x, \lambda) + r_{-3}(x, D_x, \lambda)) D_n^* (iA_2(x') f_1 \otimes \delta(x_n)) \\ &= \gamma_0 r_{-2}(x, D_x, \lambda)(iA_1(x', D_{x'}) f_1 \otimes \delta(x_n)) + \gamma_0 r_{-3}(x, D_x, \lambda)(iA_1(x', D_{x'}) f_1 \otimes \delta(x_n)) \\ &\quad + \gamma_0 r_{-2}(x, D_x, \lambda)(iA_2(x') f_1 \otimes D_n^* \delta(x_n)) + \gamma_0 r_{-3}(x, D_x, \lambda)(iA_2(x') f_1 \otimes D_n^* \delta(x_n)) \\ &= (2\pi)^{-n} \int e^{ix'\xi'} \lim_{x_n \to +0} \Big(\int_{\Gamma_{\xi'}} e^{ix_n \xi_n} r_{-2}(x', x_n, \xi', \xi_n, \lambda) \hat{\delta}(\xi_n) d\xi_n \Big) (iA_1(x', D_{x'}) f_1)^\wedge(\xi') d\xi' \\ &\quad + (2\pi)^{-n} \int e^{ix'\xi'} \lim_{x_n \to +0} \Big(\int_{\Gamma_{\xi'}} e^{ix_n \xi_n} r_{-3}(x', x_n, \xi', \xi_n, \lambda) \hat{\delta}(\xi_n) d\xi_n \Big) (iA_1(x', D_{x'}) f_1)^\wedge(\xi') d\xi' \\ &\quad + (2\pi)^{-n} \int e^{ix'\xi'} \lim_{x_n \to +0} \Big(\int_{\Gamma_{\xi'}} e^{ix_n \xi_n} r_{-2}(x', x_n, \xi', \xi_n, \lambda) (D_n^* \delta)^\wedge(\xi_n)) d\xi_n \Big) (iA_2(x') f_1)^\wedge(\xi') d\xi' \\ &\quad + (2\pi)^{-n} \int e^{ix'\xi'} \lim_{x_n \to +0} \Big(\int_{\Gamma_{\xi'}} e^{ix_n \xi_n} r_{-3}(x', x_n, \xi', \xi_n, \lambda) (D_n^* \delta)^\wedge(\xi_n) d\xi_n \Big) (iA_2(x') f_1)^\wedge(\xi') d\xi' \end{aligned}$$

$$\begin{aligned}
&= (2\pi)^{-n}\int e^{ix'\xi'}\int_{\Gamma_{\xi'}} r_{-2}(x',0,\xi',\xi_n,\lambda)d\xi_n(iA_1(x',D_{x'})f_1)^\wedge(\xi')d\xi' \\
&\quad + (2\pi)^{-n}\int e^{ix'\xi'}\int_{\Gamma_{\xi'}} r_{-3}(x',0,\xi',\xi_n,\lambda)d\xi_n(iA_1(x',D_{x'})f_1)^\wedge(\xi')d\xi' \\
&\quad + (2\pi)^{-n}\int e^{ix'\xi'}\int_{\Gamma_{\xi'}} r_{-2}(x',0,\xi',\xi_n,\lambda)\xi_n d\xi_n(iA_2(x')f_1)^\wedge(\xi')d\xi' \\
&\quad + (2\pi)^{-n}\int e^{ix'\xi'}\int_{\Gamma_{\xi'}} r_{-3}(x',0,\xi',\xi_n,\lambda)\xi_n d\xi_n(iA_2(x')f_1)^\wedge(\xi')d\xi' \\
&= (2\pi)^{1-n}\int e^{ix'\xi'}(r_1)_{-1}(x',\xi',\lambda)(iA_1(x',D_{x'})f_1)^\wedge(\xi'))d\xi' \\
&\quad + (2\pi)^{1-n}\int e^{ix'\xi'}(r_2)_{-2}(x',\xi',\lambda)(iA_1(x',D_{x'})f_1)^\wedge(\xi'))d\xi' \\
&\quad + (2\pi)^{1-n}\int e^{ix'\xi'}(r_3)_{0}(x',\xi',\lambda)(iA_2(x')f_1)^\wedge(\xi'))d\xi' \\
&\quad + (2\pi)^{1-n}\int e^{ix'\xi'}(r_4)_{-1}(x',\xi',\lambda)(iA_2(x')f_1)^\wedge(\xi'))d\xi' \\
&= (r_1)_{-1}(x',D_{x'},\lambda)iA_1(x',D_{x'})f_1 + (r_3)_0(x',D_{x'},\lambda)iA_2(x')f_1 \\
&\quad + (r_2)_{-2}(x',D_{x'},\lambda)iA_1(x',D_{x'})f_1 + (r_4)_{-1}(x',D_{x'},\lambda)iA_2(x')f_1 \\
&= (c_{00})_0(x',D_{x'},\lambda)f_1 + (c_{00})_{-1}(x',D_{x'},\lambda)f_1
\end{aligned}$$

where

$$\begin{aligned}
(D_n^*\delta)^\wedge(\xi_n) &= \langle e^{-ix_n\xi_n}, D_n^*\delta\rangle = \langle D_n e^{-ix_n\xi_n},\delta\rangle = \langle \xi_n e^{-ix_n\xi_n},\delta\rangle \\
&= \xi_n\hat{\delta}(x_n) = \xi_n,
\end{aligned}$$

$$\begin{aligned}
(r_1)_{-1} &= (2\pi)^{-1}\int_{\Gamma_{\xi'}} r_{-2}(x',0,\xi',\xi_n,\lambda)d\xi_n, \\
(r_3)_{0} &= (2\pi)^{-1}\int_{\Gamma_{\xi'}} r_{-2}(x',0,\xi',\xi_n,\lambda)\xi_n d\xi_n, \\
(r_2)_{-2} &= (2\pi)^{-1}\int_{\Gamma_{\xi'}} r_{-3}(x',0,\xi',\xi_n,\lambda)d\xi_n, \\
(r_4)_{-1} &= (2\pi)^{-1}\int_{\Gamma_{\xi'}} r_{-3}(x',0,\xi',\xi_n,\lambda)\xi_n d\xi_n,
\end{aligned}$$

and

$$\begin{aligned}
(c_{00})_0(x',D_{x'},\lambda) &= (r_1)_{-1}(x',D_{x'},\lambda)iA_1(x',D_{x'}) + (r_3)_0(x',D_{x'},\lambda)iA_2(x'), \\
(c_{00})_{-1}(x',D_{x'},\lambda) &= (r_2)_{-2}(x',D_{x'},\lambda)iA_1(x',D_{x'}) + (r_4)_{-1}(x',D_{x'},\lambda)iA_2(x').
\end{aligned}$$

If we insert (4.13) and (4.14) instead of $r_{-2}(x',0,\xi',\xi_n,\lambda)$ and $r_{-3}(x',0,\xi',\xi_n,\lambda)$, we can compute the first two terms in the asymptotic expansion of the symbols of $\underline{C^+_{\lambda,00}}$, i.e. $(c_{00})_0$ and $(c_{00})_{-1}$:

$$\begin{aligned}
(r_1)_{-1}(x',\xi',\lambda) &= \frac{1}{2\pi}\int_{\Gamma_{\xi'}} r_{-2}(x',0,\xi',\xi_n,\lambda)d\xi_n \\
&= \frac{1}{2\pi}\int_{\Gamma_{\xi'}} \frac{1}{A_2(x')(\kappa_1(x',\xi',\lambda)-\xi_n)(\kappa_2(x',\xi',\lambda)-\xi_n)}d\xi_n \\
&= \frac{A_2^{-1}(x')}{2\pi}\int_{\Gamma_{\xi'}} \frac{\frac{1}{(\kappa_2(x',\xi',\lambda)-\xi_n)}}{\kappa_1(x',\xi',\lambda)-\xi_n}d\xi_n \\
&= iA_2^{-1}(x')\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{(\kappa_2(x',\xi',\lambda)-\xi_n)}}{\kappa_1(x',\xi',\lambda)-\xi_n} \\
&= \frac{-iA_2^{-1}(x')}{\kappa_1(x',\xi',\lambda)-\kappa_2(x',\xi',\lambda)}
\end{aligned}$$

$$\begin{aligned}
(r_3)_0(x',\xi',\lambda) &= \frac{1}{2\pi}\int_{\Gamma_{\xi'}} r_{-2}(x',0,\xi',\xi_n,\lambda)\xi_n d\xi_n \\
&= \frac{1}{2\pi}\int_{\Gamma_{\xi'}} \frac{\xi_n}{A_2(x')(\kappa_1(x',\xi',\lambda)-\xi_n)(\kappa_2(x',\xi',\lambda)-\xi_n)}d\xi_n \\
&= \frac{A_2^{-1}(x)}{2\pi}\int_{\Gamma_{\xi'}} \frac{\frac{\xi_n}{(\kappa_2(x',\xi',\lambda)-\xi_n)}}{\kappa_1(x',\xi',\lambda)-\xi_n}d\xi_n \\
&= iA_2^{-1}(x')\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{(\kappa_2(x',\xi',\lambda)-\xi_n)}}{\kappa_1(x',\xi',\lambda)-\xi_n} \\
&= \frac{-iA_2^{-1}(x')\kappa_1(x',\xi',\lambda)}{\kappa_1(x',\xi',\lambda)-\kappa_2(x',\xi',\lambda)}
\end{aligned}$$

$$\begin{aligned}
(r_2)_{-2}(x',\xi',\lambda) &= \frac{1}{2\pi}\int_{\Gamma_{\xi'}} r_{-3}(x',0,\xi',\xi_n,\lambda)d\xi_n \\
&= \frac{1}{(\kappa_1-\kappa_2)^2}\Big[\frac{-i\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_1}{A_2^2(x')} + \frac{\partial_{x_n}A_2(x')}{A_2^2(x')} \\
&\quad - \frac{i\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_2}{A_2^2(x')} + \frac{\partial_{x_n}A_2(x')}{A_2^2(x')} + \frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_1}{A_2(x')}\frac{2}{\kappa_1-\kappa_2} \\
&\quad - \frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')}\frac{2}{\kappa_1-\kappa_2} + \frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_1}{A_2(x')}\frac{2}{\kappa_1-\kappa_1} \\
&\quad - \frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')}\frac{2}{\kappa_1-\kappa_2} + \frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_2}{A_2(x')}\frac{2}{\kappa_1-\kappa_2}
\end{aligned}$$

$$+\frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')}\frac{2}{\kappa_1-\kappa_2}+\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_2}{A_2(x')}\frac{1}{\kappa_1-\kappa_2}-\frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')}\frac{1}{\kappa_1-\kappa_2}$$
$$+\frac{a_1(x',\xi')}{A_2^2(x')}\frac{2}{\kappa_1-\kappa_2}\Big]=\frac{b_1}{(\kappa_1-\kappa_2)^2}$$

where $b_1(x',\xi',\lambda)\in C^\tau S^0_{1,0}$.

$$\begin{aligned}
(r_4)_{-1}(x',\xi',\lambda) &= \frac{1}{2\pi}\int_{\Gamma_{\xi'}} r_{-3}(x',0,\xi',\xi_n,\lambda)\xi_n d\xi_n\\
&= \frac{\kappa_1+\kappa_2}{(\kappa_1-\kappa_2)^2}\Big[\frac{-i\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_1}{A_2^2(x')}\frac{\kappa_2}{\kappa_1+\kappa_2}+\frac{\partial_{x_n}A_2(x')}{A_2^2(x')}\frac{\kappa_2}{\kappa_1+\kappa_2}\\
&\quad -\frac{i\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_2}{A_2^2(x')}\frac{\kappa_1}{\kappa_1+\kappa_2}+\frac{\partial_{x_n}A_2(x')}{A_2^2(x')}\frac{\kappa_1}{\kappa_1+\kappa_2}\\
&\quad +\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_1}{A_2(x')}\frac{2\kappa_2}{\kappa_1^2-\kappa_2^2}-\frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')}\frac{2\kappa_2}{\kappa_1^2-\kappa_2^2}\\
&\quad +\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_1}{A_2(x')}\frac{1}{\kappa_1-\kappa_2}-\frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')}\frac{1}{\kappa_1-\kappa_2}\\
&\quad +\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_2}{A_2(x')}\frac{1}{\kappa_1-\kappa_2}+\frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')}\frac{1}{\kappa_1-\kappa_2}\\
&\quad -\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_2}{A_2(x')}\frac{\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')}\frac{\kappa_1}{\kappa_1^2-\kappa_2^2}+\frac{a_1(x',\xi')}{A_2^2(x')}\frac{1}{\kappa_1^2-\kappa_2^2}\Big]\\
&= \frac{(\kappa_1+\kappa_2)b_2}{(\kappa_1-\kappa_2)^2},
\end{aligned}$$

where $b_2(x',\xi',\lambda)\in C^\tau S^0_{1,0}$. Hence

$$\begin{aligned}
&\sigma((c_{00})_0(x',D_{x'},\lambda))+\sigma((c_{00})_{-1}(x',D_{x'},\lambda))\\
&= \sigma((r_1)_{-1}(x',D_{x'},\lambda)iA_1(x',D_{x'}))+\sigma((r_3)_0(x',D_{x'},\lambda)iA_2(x'))\\
&\quad +\sigma((r_2)_{-2}(x',D_{x'},\lambda)iA_1(x',D_{x'}))+\sigma((r_4)_{-1}(x',D_{x'},\lambda)iA_2(x'))\\
&= (r_1)_{-1}(x',\xi',\lambda)\sharp_1 ia_1(x',\xi')+(r_3)_0(x',\xi',\lambda)\sharp_1 iA_2(x')\\
&\quad +(r_2)_{-2}(x',\xi',\lambda)\sharp_0 ia_1(x',\xi')+(r_4)_{-1}(x',\xi',\lambda)\sharp_0 iA_2(x')
\end{aligned}$$

$$
\begin{aligned}
= \quad & (r_1)_{-1}(x',\xi',\lambda) i a_1(x',\xi') + \sum_{j=1}^{n-1} D_{\xi_j}(r_1)_{-1}(x',\xi',\lambda)\partial_{x_j} i a_1(x',\xi') + \tilde{\mathcal{R}}_1 \\
& + i(r_3)_0(x',\xi',\lambda)A_2(x') + i\sum_{j=1}^{n-1} D_{\xi_j}(r_3)_0(x',\xi',\lambda)\partial_{x_j}A_2(x') + \tilde{\mathcal{R}}_2 \\
& + i(r_2)_{-2}(x',\xi',\lambda)a_1(x',\xi') + \tilde{\mathcal{R}}_3 \\
& + i(r_4)_{-1}(x',\xi',\lambda)A_2(x') + \tilde{\mathcal{R}}_4 \\
= \quad & \frac{A_2^{-1}(x')a_1(x',\xi')}{\kappa_1-\kappa_2} + \sum_{j=1}^{n-1} D_{\xi_j}(\frac{A_2^{-1}(x')}{\kappa_1-\kappa_2})\partial_{x_j}a_1(x',\xi') + \tilde{\mathcal{R}}_1 \\
& + \frac{\kappa_1}{\kappa_1-\kappa_2} + \sum_{j=1}^{n-1} D_{\xi_j}(\frac{A_2^{-1}(x')\kappa_1}{\kappa_1-\kappa_2})\partial_{x_j}A_2(x') + \tilde{\mathcal{R}}_2 + \frac{ib_1a_1(x',\xi')}{(\kappa_1-\kappa_2)^2} + \tilde{\mathcal{R}}_3 \\
& + \frac{i(\kappa_1+\kappa_2)b_2A_2(x')}{(\kappa_1-\kappa_2)^2} + \tilde{\mathcal{R}}_4,
\end{aligned}
$$

where from Theorem 2.1.18

$$\tilde{\mathcal{R}}_1 \quad : \quad H^{s-\theta_1}(\partial D) \to H^s(\partial D)$$

is bounded for $\theta_1 \in (1,2)$, $-(\tau-1) < s < \tau-1$ and $-(\tau-\theta_1) < s-1 < \tau$,

$$\tilde{\mathcal{R}}_2 \quad : \quad H^{s-2-\theta_2}(\partial D) \to H^s(\partial D)$$

is bounded for $\theta_2 \in (1,2)$, $-(\tau-1) < s < \tau-1$ and $-(\tau-\theta_2) < s < \tau$.

$$\tilde{\mathcal{R}}_3 \quad : \quad H^{s-1-\theta_3}(\partial D) \to H^s(\partial D)$$

is bounded for $\theta_3 \in (0,1)$, $-\tau < s < \tau$ and $-(\tau-\theta_3) < s-2 < \tau$, and

$$\tilde{\mathcal{R}}_4 \quad : \quad H^{s-1-\theta_4}(\partial D) \to H^s(\partial D)$$

is bounded for $\theta_4 \in (0,1)$, $-\tau < s < \tau$ and $-(\tau-\theta_4) < s-1 < \tau$.
Consequently we have

$$
\begin{aligned}
\sigma((c_{00})_0(x',D_{x'},\lambda)) \quad &= \quad \frac{A_2^{-1}(x')a_1(x',\xi')}{\kappa_1-\kappa_2} + \frac{\kappa_1}{\kappa_1-\kappa_2} = \frac{\kappa_1}{\kappa_1-\kappa_2}\Big(\frac{a_1(x',\xi')}{A_2(x')\kappa_1}+1\Big) \\
&= \quad \frac{-\kappa_2}{\kappa_1-\kappa_2} \in C^\tau S^0_{1,0}. \\
\sigma((c_{00})_{-1}(x',D_{x'})) \quad &= \quad A_2^{-1}(x')\sum_{j=1}^{n-1} D_{\xi_j}(\frac{1}{\kappa_1-\kappa_2})\partial_{x_j}a_1(x',\xi') \\
& \quad + A_2^{-1}(x')\sum_{j=1}^{n-1} D_{\xi_j}(\frac{\kappa_1}{\kappa_1-\kappa_2})\partial_{x_j}A_2(x') \\
& \quad + \frac{ib_1a_1(x',\xi')}{(\kappa_1-\kappa_2)^2} + \frac{i(\kappa_1+\kappa_2)b_2A_2(x')}{(\kappa_1-\kappa_2)^2} \in C^{\tau-1}S^{-1}_{1,0}.
\end{aligned}
$$

$$
\begin{aligned}
\underline{C^+_{\lambda,01}} f_2 &= \gamma_0 \underline{Q_\lambda} \gamma_0^* i A_2(x') f_2 \\
&= \gamma_0 (r_{-2}(x, D_x, \lambda) + r_{-3}(x, D_x, \lambda))(iA_2(x') f_2 \otimes \delta(x_n)) \\
&= \gamma_0 r_{-2}(x, D_x, \lambda)(iA_2(x') f_2 \otimes \delta(x_n)) + \gamma_0 r_{-3}(x, D_x, \lambda)(iA_2(x') f_2 \otimes \delta(x_n)) \\
&= (2\pi)^{-n} \gamma_0 \int e^{ix'\xi'} \Big(\int_{\Gamma_{\xi'}} e^{ix_n \xi_n} r_{-2}(x', x_n, \xi', \xi_n, \lambda) \hat{\delta}(x_n) d\xi_n \Big) (iA_2(x') f_1)^\wedge(\xi') d\xi' \\
&\quad + (2\pi)^{-n} \gamma_0 \int e^{ix'\xi'} \Big(\int_{\Gamma_{\xi'}} e^{ix_n \xi_n} r_{-3}(x', x_n, \xi', \xi_n, \lambda) \hat{\delta}(x_n) d\xi_n \Big) (iA_2(x') f_1)^\wedge(\xi') d\xi' \\
&= (2\pi)^{-n} \int e^{ix'\xi'} \lim_{x_n \to +0} \Big(\int_{\Gamma_{\xi'}} e^{ix_n \xi_n} r_{-2}(x', x_n, \xi', \xi_n, \lambda) d\xi_n \Big) (iA_2(x') f_1)^\wedge(\xi') d\xi' \\
&\quad + (2\pi)^{-n} \int e^{ix'\xi'} \lim_{x_n \to +0} \Big(\int_{\Gamma_{\xi'}} e^{ix_n \xi_n} r_{-3}(x', x_n, \xi', \xi_n, \lambda) d\xi_n \Big) (iA_2(x') f_1)^\wedge(\xi') d\xi' \\
&= (2\pi)^{-n} \int e^{ix'\xi'} \int_{\Gamma_{\xi'}} r_{-2}(x', 0, \xi', \xi_n, \lambda) d\xi_n (iA_2(x') f_1)^\wedge(\xi') d\xi' \\
&\quad + (2\pi)^{-n} \int e^{ix'\xi'} \int_{\Gamma_{\xi'}} r_{-3}(x', x_n, \xi', \xi_n, \lambda) d\xi_n (iA_2(x') f_1)^\wedge(\xi') d\xi' \\
&= (2\pi)^{1-n} \int e^{ix'\xi'} (r_1)_{-1}(x', 0, \xi', \xi_n, \lambda)(iA_2(x') f_1)^\wedge(\xi') d\xi' \\
&\quad + (2\pi)^{1-n} \int e^{ix'\xi'} (r_2)_{-2}(x', 0, \xi', \xi_n, \lambda)(iA_2(x') f_1)^\wedge(\xi') d\xi' \\
&= r_{-1}(x', D_{x'}, \lambda)(iA_2(x') f_2) + r_{-2}(x', D_{x'}, \lambda)(iA_2(x') f_2) \\
&= (c_{01})_{-1}(x', D_{x'}, \lambda) f_2 + (c_{01})_{-2}(x', D_{x'}, \lambda) f_2
\end{aligned}
$$

where

$$
\begin{aligned}
(r_1)_{-1} &= (2\pi)^{-1} \int_{\Gamma_{\xi'}} r_{-2}(x', 0, \xi', \xi_n, \lambda) d\xi_n = \frac{-iA_2^{-1}(x')}{\kappa_1 - \kappa_2}, && (4.16) \\
(r_2)_{-2} &= (2\pi)^{-1} \int_{\Gamma_{\xi'}} r_{-3}(x', 0, \xi', \xi_n, \lambda) d\xi_n = \frac{b_1}{(\kappa_1 - \kappa_2)^2}, && (4.17)
\end{aligned}
$$

and

$$
\begin{aligned}
(c_{01})_{-1} &= (r_1)_{-1}(x', D_{x'}, \lambda)(iA_2(x')), \\
(c_{01})_{-2} &= (r_2)_{-2}(x', D_{x'}, \lambda)(iA_2(x')).
\end{aligned}
$$

If we insert (4.16) and (4.17), we can compute the first two terms in the asymptotic expansion of the symbols of $\underline{C^+_{\lambda,01}}$, i.e. $(c_{01})_{-1}$ and $(c_{01})_{-2}$:

$$\begin{aligned}
&\sigma((c_{01})_{-1}(x', D_{x'}, \lambda)) + \sigma((c_{01})_{-2}(x', D_{x'}, \lambda)) \\
&\quad= \sigma((r_1)_{-1}(x', D_{x'}, \lambda)(iA_2(x'))) + \sigma((r_2)_{-2}(x', D_{x'}, \lambda)(iA_2(x'))) \\
&\quad= (r_1)_{-1}(x', \xi', \lambda)\sharp_1 iA_2(x')) + (r_2)_{-2}(x', \xi', \lambda)\sharp_0 iA_2(x') \\
&\quad= \frac{1}{\kappa_1 - \kappa_2} + \sum_{j=1}^{n-1} D_{\xi_j}(\frac{A_2^{-1}(x')}{\kappa_1 - \kappa_2})\partial_{x_j} A_2(x') + \tilde{\tilde{\mathcal{R}}}_1 \\
&\quad\quad - i\frac{b_1 A_2(x')}{(\kappa_1 - \kappa_2)^2} + \tilde{\tilde{\mathcal{R}}}_2 \\
&\quad= \frac{1}{\kappa_1 - \kappa_2} + A_2^{-1}(x')\sum_{j=1}^{n-1} D_{\xi_j}(\frac{1}{\kappa_1 - \kappa_2})\partial_{x_j} A_2(x') \\
&\quad\quad + \frac{ib_1 A_2(x')}{(\kappa_1 - \kappa_2)^2} + \tilde{\tilde{\mathcal{R}}}_1 + \tilde{\tilde{\mathcal{R}}}_2.
\end{aligned}$$

We conclude that

$$\sigma((c_{01})_{-1}(x', D_{x'}, \lambda)) = \frac{1}{\kappa_1 - \kappa_2} \in C^\tau S^{-1}_{1,0} \tag{4.18}$$

$$\begin{aligned}
\sigma((c_{01})_{-2}(x', D_{x'}, \lambda)) &= A_2^{-1}(x')\sum_{j=1}^{n-1} D_{\xi_j}(\frac{1}{\kappa_1 - \kappa_2})\partial_{x_j} A_2(x') \\
&\quad + \frac{ib_1 A_2(x')}{(\kappa_1 - \kappa_2)^2} \in C^{\tau-1} S^{-2}_{1,0}.
\end{aligned} \tag{4.19}$$

and from Theorem 2.1.18

$$\tilde{\tilde{\mathcal{R}}}_1 \quad : \quad H^{s-1-\theta_5}(\partial D) \to H^s(\partial D)$$

is bounded for $\theta_5 \in (1, 2)$, $-(\tau - 1) < s < \tau - 1$ and $-(\tau - \theta_5) < s < \tau$, and

$$\tilde{\tilde{\mathcal{R}}}_2 \quad : \quad H^{s-2-\theta_6}(\partial D) \to H^s(\partial D)$$

is bounded for $\theta_6 \in (0, 1)$, $-\tau < s < \tau$ and $-(\tau - \theta_6) < s < \tau$.
Summing up, we have shown:

Lemma 4.1.1. *In the* 2×2*-matrix of pseudodifferential operator* C^+_λ *in* (4.15)*, the first two terms in the asymptotic expansion of the symbols* $C^+_{\lambda,00}$ *and* $C^+_{\lambda,01}$ *are computed from the two roots of the principal symbol of differential operator* $A - \lambda$ *in* (4.7) *and they have the following form:*

$$\begin{aligned}
\sigma(\underline{C^+_{\lambda,00}}) &= \overbrace{(c_{00})_0}^{C^\tau S^0_{1,0}} + \overbrace{(c_{00})_{-1}}^{C^{\tau-1}S^{-1}_{1,0}} \\
&= -\kappa_2(\kappa_1-\kappa_2)^{-1} + A_2^{-1}(x')\sum_{j=1}^{n-1} D_{\xi_j}(\frac{1}{\kappa_1-\kappa_2})\partial_{x_j} a_1(x',\xi') \\
&\quad + A_2^{-1}(x')\sum_{j=1}^{n-1} D_{\xi_j}(\frac{\kappa_1}{\kappa_1-\kappa_2})\partial_{x_j} A_2(x') + \frac{ib_1a_1(x',\xi')}{(\kappa_1-\kappa_2)^2} + \frac{i(\kappa_1+\kappa_2)b_2A_2(x')}{(\kappa_1-\kappa_2)^2} \\
&\in C^{\tau-1}S^0_{1,0}, \\
\sigma(\underline{C^+_{\lambda,01}}) &= \overbrace{(c_{01})_{-1}}^{C^\tau S^{-1}_{1,0}} + \overbrace{(c_{01})_{-2}}^{C^{\tau-1}S^{-2}_{1,0}} \\
&= (\kappa_1-\kappa_2)^{-1} + A_2^{-1}(x')\sum_{j=1}^{n-1} D_{\xi_j}(\frac{1}{\kappa_1-\kappa_2})\partial_{x_j} A_2(x') + \frac{ib_1A_2(x')}{(\kappa_1-\kappa_2)^2} \\
&\in C^{\tau-1}S^{-1}_{1,0}.
\end{aligned}$$

4.2 The Dirichlet-to-Neumann Operator and the Operator T(λ)

This leads to the construction of the Dirichlet-to-Neumann operator

$$\Lambda(\lambda) = \gamma_1 K^D(\lambda): \tag{4.20}$$

Let $\varphi = \{\varphi_0, \varphi_1\} \in \rho(N(A-\lambda))$ satisfy

$$\varphi_1 = \Lambda(\lambda)\varphi_0. \tag{4.21}$$

Then $C^+_\lambda \varphi = \varphi$, i.e.

$$\begin{pmatrix} C^+_{\lambda,00} & C^+_{\lambda,01} \\ C^+_{\lambda,10} & C^+_{\lambda,11} \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}. \tag{4.22}$$

Then an insertion of (4.21) in (4.22) gives

$$C^+_{\lambda,00} + C^+_{\lambda,01}\Lambda(\lambda) = I$$

or

$$C^+_{\lambda,01}\Lambda(\lambda) = I - C^+_{\lambda,00}.$$

Since $C^+_{\lambda,01}$ is elliptic, we can construct a rough parametrix $P_{01} = Op((p_{01})_1) + Op((p_{01})_0)$. By (4.18) and (4.19) we have

$$\begin{aligned}
(p_{01})_1 &= \frac{1}{(c_{01})_{-1}} = \kappa_1 - \kappa_2 \in C^\tau S^1_{1,0} \\
(p_{01})_0 &= -(p_{01})_1 \sum_{j=1}^{n-1} D_{\xi_j}(c_{01})_{-1}\partial_{x_j}(p_{01})_1 - (p_{01})_1(c_{01})_{-2}(p_{01})_1 \in C^{\tau-1}S^0_{1,0}.
\end{aligned}$$

Clearly

$$(p_{01})_1(x', D_{x'}, \lambda) : H^{s+1}(\partial D) \to H^s(\partial D)$$

when $-\tau < s < \tau$, and

$$(p_{01})_0(x', D_{x'}, \lambda) : H^s(\partial D) \to H^s(\partial D)$$

when $-\tau + 1 < s < \tau - 1$ are continuous. Therefore from Theorem 2.1.18

$$\tilde{\mathcal{R}}_{\theta_7}(x', D_{x'}) = P_{01}(x', D_{x'})C^+_{\lambda,01}(x', D_{x'}) - I : H^{s-\theta_7}(\partial D) \to H^s(\partial D)$$

for $\theta_7 \in (1, 2)$, $-\tau + 2 < s < \tau - 2$, $-\tau + 1 + \theta_7 < s + 1 < \tau - 1$, and

$$\tilde{\tilde{\mathcal{R}}}_{\theta_7}(x', D_{x'}) = C^+_{\lambda,01}(x', D_{x'})P_{01}(x', D_{x'}) - I : H^{s-\theta_7}(\partial D) \to H^s(\partial D)$$

for $\theta_7 \in (1, 2)$, $-\tau + 2 < s < \tau - 2$, $-\tau + 1 + \theta_7 < s - 1 < \tau - 1$ are bounded. Hence the fact that

$$P_{01}(I - C^+_{\lambda,00}) = P_{01}(C^+_{\lambda,01}\Lambda(\lambda)) = (I + \tilde{\mathcal{R}}_{\theta_3})\Lambda(\lambda)$$

implies that

$$\Lambda(\lambda) = P_{01}(I - C^+_{\lambda,00}) - \tilde{\mathcal{R}}_{\theta_7}(\lambda)\Lambda(\lambda),$$

and since $\Lambda(\lambda)$ has order one and $\tilde{\mathcal{R}}_{\theta_7}$ has negative order less than -1, $\tilde{\mathcal{R}}_{\theta_7}\Lambda(\lambda)$ has negative order. Therefore we get

$$\begin{aligned}
\Lambda(\lambda) &= P_{01}(I - C^+_{\lambda,00})) - \tilde{\mathcal{R}}_{\theta_7}(\lambda)\Lambda(\lambda) \\
&= Op\Big((p_{01})_1 + (p_{01})_0\Big)Op\Big((1 - (c_{00})_0 - (c_{00})_{-1}) + \overbrace{\mathcal{R}'}^{\text{order}<-1}\Big) - \tilde{\mathcal{R}}_{\theta_7}(\lambda)\Lambda(\lambda) \\
&= (p_{01})_1\sharp_1(1 - (c_{00})_0)(x', D_{x'}, \lambda) + \tilde{\tilde{\tilde{\mathcal{R}}}}_1 - (p_{01})_1\sharp_0(q_{00})_{-1}(x', D_{x'}, \lambda) + \tilde{\tilde{\tilde{\mathcal{R}}}}_2 \\
&\quad - (p_{01})_0\sharp_0(1 - (c_{00})_0)(x', D_{x'}, \lambda) + \tilde{\tilde{\tilde{\mathcal{R}}}}_3 - (p_{01})_0\sharp_0(c_{00})_{-1}(x', D_{x'}, \lambda) + \tilde{\tilde{\tilde{\mathcal{R}}}}_4 \\
&\quad + \overbrace{Op((p_{01})_1 + (p_{01})_0)\mathcal{R}'}^{\text{order}<0} - \tilde{\mathcal{R}}_{\theta_7}(\lambda)\Lambda(\lambda) \\
&= (p_{01})_1(1 - (c_{00})_0) + \sum_{j=1}^{n-1} D_{\xi_j}(p_{01})_1\partial_{x_j}(1 - (c_{00})_0) + \tilde{\tilde{\tilde{\mathcal{R}}}}_1 \\
&\quad - (p_{01})_1(c_{00})_{-1} + \tilde{\tilde{\tilde{\mathcal{R}}}}_2 + (p_{01})_0(1 - (c_{00})_0) + \tilde{\tilde{\tilde{\mathcal{R}}}}_3 \\
&\quad - (p_{01})_0(c_{00})_{-1} + \tilde{\tilde{\tilde{\mathcal{R}}}}_4 + \overbrace{Op((p_{01})_1 + (p_{01})_0)\mathcal{R}'}^{\text{order}<0} - \tilde{\mathcal{R}}_{\theta_7}(\lambda)\Lambda(\lambda) \\
&= \lambda_1 - \lambda_0 + \tilde{\tilde{\mathcal{R}}}
\end{aligned}$$

where from Theorem 2.1.18

$$\tilde{\tilde{\tilde{\mathcal{R}}}}_1 \quad : \quad H^{s+1-\theta_8}(\partial D) \to H^s(\partial D)$$

is bounded for $\theta_8 \in (1, 2)$, $-(\tau - 1) < s < \tau - 1$ and $-(\tau - \theta_8) < s + 1 < \tau$,

$$\tilde{\tilde{\tilde{\mathcal{R}}}}_2 \quad : \quad H^{s-\theta_9}(\partial D) \to H^s(\partial D)$$

is bounded for $\theta_9 \in (0,1)$, $-(\tau-1) < s < \tau-1$ and $-(\tau-1-\theta_9) < s+1 < \tau-1$.

$$\tilde{\tilde{\tilde{\mathcal{R}}}}_3 \quad : \quad H^{s-\theta_{10}}(\partial D) \to H^s(\partial D)$$

is bounded for $\theta_{10} \in (0,1)$, $-(\tau-1) < s < \tau-1$ and $-(\tau-\theta_{10}) < s < \tau$, and

$$\tilde{\tilde{\tilde{\mathcal{R}}}}_4 \quad : \quad H^{s-1-\theta_{11}}(\partial D) \to H^s(\partial D)$$

is bounded for $\theta_{11} \in (0,1)$, $-(\tau-1) < s < \tau-1$ and $-(\tau-1-\theta_{11}) < s-1 < \tau-1$.
Therefore $\tilde{\tilde{\mathcal{R}}}$ has order less than zero.
Summing up, we have shown:

Lemma 4.2.1. *Let $\Lambda(\lambda)$ be as in* (4.20) *the Dirichlet-to-Neumann operator. The first two terms in the asymptotic expansion of $\Lambda(\lambda)$ are computed from first two terms in the asymptotic expansion of the symbols $C^+_{\lambda,00}$ and $C^+_{\lambda,01}$ of the Calderón projector C^+_λ and they have the following form:*

$$\begin{cases} \lambda_1 &= (p_{01})_1(1-(c_{00})_0) = (\kappa_1-\kappa_2)(1+\frac{\kappa_2}{\kappa_1-\kappa_2}) = \kappa_1 \in C^\tau S^1_{1,0} \\ \lambda_0 &= \sum_{j=1}^{n-1} D_{\xi_j}(p_{01})_1\partial_{x_j}(1-(c_{00})_0) - (p_{01})_1(c_{00})_{-1} - (p_{01})_0(1-(c_{00})_0) \\ &\in C^{\tau-1}S^0_{1,0}. \end{cases} \tag{4.23}$$

We next study the operator $T(\lambda)$.

$$\begin{aligned} T(\lambda) : D(T(\lambda)) \subset H^{s+\frac{3}{2}}(\partial D) &\to H^{s+\frac{3}{2}}(\partial D) \\ \psi &\longmapsto LK^D(\lambda)(\psi), \end{aligned}$$

with $D(T(\lambda)) = \left\{\psi \in H^{s+\frac{3}{2}}(\partial D); T(\lambda)\psi \in H^{s+\frac{3}{2}}(\partial D)\right\}$. Then

$$T(\lambda) = \mu_1(x')\Lambda(\lambda) + \mu_2(x') \tag{4.24}$$

is a pseudodifferential operator of first order on the boundary ∂D, and its symbol $t(x',\xi',\lambda)$ has an asymptotic expansion

$$\begin{aligned} t(x',\xi',\lambda) &= t_1(x',\xi',\lambda) + t_0(x',\xi',\lambda) + ... \\ &= [i\mu_1(x')\lambda_1(x',\xi',\lambda) + \mu_2(x')] + i[\mu_1(x')\lambda_0(x',\xi',\lambda)] \\ &\quad + \text{ terms of negative order}. \end{aligned}$$

Fix $\tilde{\lambda}$. We will see that there exists $C \geq 0, c > 0$ such that $t_1(x',\xi',\tilde{\lambda}) + t_0(x',\xi',\tilde{\lambda}) + \tilde{\mu}$ is invertible for $\{|\tilde{\mu}| < c\}$ or $\tilde{\mu}$ outside a suitable sector contained around the positive real axis for $|\xi'| \geq C$. Moreover, (t_1+t_0) is hypoelliptic in the sense of Definition (2.5.1).

1) First we show that $|t_1(x',\xi',\tilde{\lambda}) + t_0(x',\xi',\tilde{\lambda})| \geq C'$ for large ξ' (C' is a positive constant):

$$\begin{aligned} |t_1(x',\xi',\tilde{\lambda}) + t_0(x',\xi',\tilde{\lambda})| &= |i\mu_1(x')(\lambda_1(x',\xi',\tilde{\lambda}) + \lambda_0(x',\xi',\tilde{\lambda})) + \mu_2(x')| \\ &\geq \left|\mathrm{Re}\left(i\mu_1(x')(\lambda_1(x',\xi',\tilde{\lambda}) + \lambda_0(x',\xi',\tilde{\lambda}))\right) + \mu_2(x')\right| \\ &\overbrace{\geq}^{(4.9)} |\mu_1(x')(p_1(x',\xi',\tilde{\lambda}) + \mathrm{Re}(i\lambda(x',\xi',\tilde{\lambda})) + \mu_2(x')| \\ &\geq C_{\overline{D}}\mu_1(x')|\xi'| + \mu_2(x') \text{ for } |\xi'| \geq C. \end{aligned} \tag{4.25}$$

(4.25) is true, since $|p_1(x',\xi',\tilde\lambda)+\mathrm{Re}(i\lambda_0(x',\xi',\tilde\lambda))|\geq C_{\overline{D}}|\xi'|$, for $|\xi'|\geq C$. Then for $C_{\overline{D}}\geq C'$,

$$|t_1(x',\xi',\tilde\lambda)+t_0(x',\xi',\tilde\lambda)|\geq C_{\overline{D}}\mu_1(x')|\xi'|+\mu_2(x')\geq C'(\mu_1(x')|\xi'|+1). \quad (4.26)$$

2) We show that for all $\alpha\in\mathbb{N}_0$ and $|\beta|\leq[\tau]-1$

$$\frac{\left|D^\alpha_{\xi'}\partial^\beta_{x'}(t_1(x',\xi',\tilde\lambda)+t_0(x',\xi',\tilde\lambda))\right|}{\left|t_1(x',\xi',\tilde\lambda)+t_0(x',\xi',\tilde\lambda)\right|}\leq C(1+|\xi'|)^{-|\alpha|+\frac{|\beta|}{2}}. \quad (4.27)$$

For $\alpha=\beta=0$ this is clear. According to Leibniz product, $D^\alpha_{\xi'}\partial^\beta_{x'}(t_1+t_0)$ is a linear combination of terms of the form

$$\partial^{\beta_1}_{x'}\mu_1 D^\alpha_{\xi'}\partial^{\beta_2}_{x'}\lambda_1+\partial^{\beta_1}_{x'}\mu_1 D^\alpha_{\xi'}\partial^{\beta_2}_{x'}\lambda_0+\partial^\beta_{x'}\mu_2,\ |\beta_1|+|\beta_2|=|\beta|\,,$$

and $D^\alpha_{\xi'}\partial^{\beta_2}_{x'}\lambda_1=O(\langle\xi\rangle^{1-|\alpha|})$, $D^\alpha_{\xi'}\partial^{\beta_2}_{x'}\lambda_0=O(\langle\xi\rangle^{-|\alpha|})$. Let $|\alpha|=1$.

$$\begin{aligned}|D_{\xi'}(t_1(x',\xi',\tilde\lambda)+t_0(x',\xi',\tilde\lambda))| &\leq |i\mu_1D_{\xi'}\lambda_1|+|i\mu_1D_{\xi'}\lambda_0|\\ &\leq C(\mu_1+|\xi'|^{-1})\leq C'(\mu_1|\xi'|+1)(1+|\xi'|)^{-1}\\ &\leq C'|t(x',\xi',\tilde\lambda)-\tilde\mu|(1+|\xi'|)^{-1}.\end{aligned}$$

If we assume that $\tau\geq 4$, then, since μ_1 is a nonnegative C^τ function on ∂D, we have for $|\gamma|=1\leq[\tau]-1$

$$\left|\partial^\gamma_{x'}\mu_1(x')\right|\leq\sqrt{2}\sqrt{\left\|\partial^{\gamma+e_j}_{x'}\mu_1\right\|_\infty}\sqrt{\partial^{\gamma-e_j}_{x'}\mu_1(x')} \quad (4.28)$$

[20, Lemma 4.3].
Let $|\beta|=1$. Then

$$\begin{aligned}&\left|\partial_{x'}(t_1(x',\xi',\tilde\lambda)+t_0(x',\xi',\tilde\lambda))\right|\\
&= \left|i\partial_{x'}(\mu_1(x')(\lambda_1(x',\xi',\tilde\lambda))+i\partial_{x'}(\mu_1(x')(\lambda_0(x',\xi',\tilde\lambda))+\partial_{x'}\mu_2(x')\right|\\
&\leq C_1|\partial_{x'}\mu_1|\,|\xi'|+C_2|\mu_1|+C_3\\
&\leq C\Big[|\partial_{x'}\mu_1|\,|\xi'|+|\mu_1|+1\Big]\\
&\overbrace{\leq}^{(4.28)} C'\Big[\sqrt{\mu_1(x')}\sqrt{\|\partial^2_{x'}\mu_1\|_{L^\infty}}|\xi'|+\mu_1(x')|\xi'|+1\Big]\\
&\leq C'\Big[\|\partial^2_{x'}\mu_1\|^{\frac12}_{L^\infty}|\xi'|^{\frac12}(\mu_1(x')|\xi'|+1)^{\frac12}+(\mu_1(x')|\xi'|+1)\Big]\\
&\overbrace{\leq}^{(4.26)} C'\Big[\|\partial^2_{x'}\mu_1\|^{\frac12}_{L^\infty}|\xi'|^{\frac12}|t(x',\xi',\tilde\lambda)-\tilde\mu|^{\frac12}+|t(x',\xi',\tilde\lambda)-\tilde\mu|\Big]\\
&= C'|t(x',\xi',\tilde\lambda)-\tilde\mu|\Big(|\xi'|^{\frac12}|t(x',\xi',\tilde\lambda)-\tilde\mu|^{-\frac12}+1\Big)\\
&\leq C''|t(x',\xi',\tilde\lambda)-\tilde\mu|(1+|\xi'|)^{\frac12}.\end{aligned}$$

For $2\leq|\beta|\leq[\tau]-1$ assertion (4.27) is trivial. Therefore

$$\frac{|D^\alpha_{\xi'}\partial^\beta_{x'}(t_1(x',\xi',\tilde\lambda)+t_0(x',\xi',\tilde\lambda))|}{|t_1(x',\xi',\tilde\lambda)+t_0(x',\xi',\tilde\lambda)|}\leq C(1+|\xi'|)^{-|\alpha|+\frac{|\beta|}{2}},\ \alpha\in\mathbb{N}_0\ ,\ |\beta|\leq[\tau]-1.$$

3) Finally we show that $\left\|(D^{\alpha}_{\xi'}\partial^{\beta}_{x'}(t_1+t_0))(t_1+t_0)^{-1}\right\|_{C^{\tau-|\beta|}} \leq \langle\xi'\rangle^{-|\alpha|+\frac{|\beta|}{2}+\frac{\tau-|\beta|}{2}}$ for $\tau\in\mathbb{Z}_{\geq 0}$ and $\tau\geq 4$.

$$\begin{aligned}
&\left\|(D^{\alpha}_{\xi'}\partial^{\beta}_{x'}(t_1+t_0))(t_1+t_0)^{-1}\right\|_{C^{\tau-|\beta|}}\\
&\leq C\sum_{|\gamma|\leq\tau-|\beta|}\left\|D^{\gamma}_{x'}\Big((D^{\alpha}_{\xi'}\partial^{\beta}_{x'}(t_1+t_0))(t_1+t_0)^{-1}\Big)\right\|_{L^\infty}\\
&= C\sum_{\substack{\tilde\beta\leq\gamma\\|\gamma|\leq\tau-|\beta|}}\left\|(D^{\alpha}_{\xi'}\partial^{\beta+\gamma-\tilde\beta}_{x'}(t_1+t_0))\partial^{\tilde\beta}_{x'}(t_1+t_0)^{-1}\right\|_{L^\infty}\\
&= C\sum_{\substack{\tilde\beta\leq\gamma\\|\gamma|\leq\tau-|\beta|\\\tilde\beta_1+\ldots+\tilde\beta_n=\tilde\beta}}\Big\|\Big(D^{\alpha}_{\xi'}\partial^{\beta+\gamma-\tilde\beta}_{x'}(t_1+t_0)\Big)\Big((t_1+t_0))^{-1}(\partial^{\tilde\beta_1}_{x'}(t_1+t_0))(t_1+t_0))^{-1}\\
&\quad\ldots(t_1+t_0))^{-1}(\partial^{\tilde\beta_n}_{x'}(t_1+t_0))(t_1+t_0))^{-1}\Big)\Big\|_{L^\infty}\\
&\leq C\sum_{\substack{\tilde\beta\leq\gamma\\|\gamma|\leq\tau-|\beta|\\\tilde\beta_1+\ldots+\tilde\beta_n=\tilde\beta}}\left\|(D^{\alpha}_{\xi'}\partial^{\beta+\gamma-\tilde\beta}_{x'}(t_1+t_0))(t_1+t_0))^{-1}\right\|_{L^\infty}\left\|(\partial^{\tilde\beta_1}_{x'}(t_1+t_0))(t_1+t_0))^{-1}\right\|_{L^\infty}\\
&\quad\ldots\left\|(\partial^{\tilde\beta_n}_{x'}(t_1+t_0))(t_1+t_0))^{-1}\right\|_{L^\infty}\\
&\leq C'\langle\xi'\rangle^{-|\alpha|+\frac{|\beta|}{2}+\frac{\tau-|\beta|}{2}}.
\end{aligned}$$

Now we have

(1) $\left|t_1(x',\xi',\tilde\lambda)+t_0(x',\xi',\tilde\lambda)+\tilde\mu\right|^{-1}\leq C\,,$

(2) $\left|D^{\alpha}_{\xi'}D^{\beta}_{x'}(t_1(x',\xi',\tilde\lambda)+t_0(x',\xi',\tilde\lambda))\right|\leq C|t_1(x',\xi',\tilde\lambda)+t_0(x',\xi',\tilde\lambda)+\tilde\mu|(1+|\xi'|)^{-|\alpha|+\frac{|\beta|}{2}}\,,$

(3) $\left\|(D^{\alpha}_{\xi'}\partial^{\beta}_{x'}(t_1+t_0))\ (t_1+t_0+\tilde\mu)^{-1}\right\|_{C^{\tau-|\beta|}}\leq C'\langle\xi'\rangle^{-|\alpha|+\frac{|\beta|}{2}+\frac{\tau-|\beta|}{2}},\ \tau\in\mathbb{Z}_{\geq 0}$ and $\tau\geq 4\,.$

We construct a parametrix to $t_1(x',\xi',\lambda)+t_0(x',\xi',\lambda)+\tilde\mu\in C^{\tau-1}S^1_{1,0}$:

$$\begin{aligned}
b_0(x',\xi',\lambda,\tilde\mu) &= (t_1(x',\xi',\lambda)+t_0(x',\xi',\lambda)+\tilde\mu)^{-1}\\
&= (i\mu_1(x')(\lambda_1+\lambda_0)+\mu_2(x')+\tilde\mu)^{-1}\in C^{\tau-1}S^0_{1,\frac12}\,.
\end{aligned}$$

From Theorem 2.5.4 $t_1+t_0+\tilde\mu\in C^{\tau-1}S^1_{1,0}$ is hypoelliptic for $\delta'=\frac12$. Then b_0 is the right and left parametrices such that:

$$\begin{aligned}
(t_1+t_0+\tilde\mu)\sharp_2 b_0 &= (t_1+t_0+\tilde\mu)b_0+D_{\xi'}(t_1+t_0+\tilde\mu)\partial_{x'}b_0\\
&\quad+\sum_{|\alpha|=2}\frac{1}{\alpha!}D^{\alpha}_{\xi'}b_0\partial^{\alpha}_{x'}(t_1+t_0+\tilde\mu)\\
&= 1+\overbrace{D_{\xi'}(t_1+t_0+\tilde\mu)\partial_{x'}b_0}^{C^{\tau-2}S^{-\frac12}_{1,\frac12}}+\overbrace{\sum_{|\alpha|=2}\frac{1}{\alpha!}D^{\alpha}_{\xi'}b_0\partial^{\alpha}_{x'}(t_1+t_0+\tilde\mu)}^{C^{\tau-3}S^{-\frac32}_{1,\frac12}}\,.
\end{aligned}$$

$$\begin{aligned}
b_0\sharp_2(t_1+t_0+\tilde{\mu}) &= b_0(t_1+t_0+\tilde{\mu}) + D_{\xi'}b_0\partial_{x'}(t_1+t_0+\tilde{\mu}) \\
&\quad + \sum_{|\alpha|=2}\frac{1}{\alpha!}D^{\alpha}_{\xi'}b_0\partial^{\alpha}_{x'}(t_1+t_0+\tilde{\mu}) \\
&= 1 + \overbrace{D_{\xi'}b_0\partial_{x'}(t_1+t_0+\tilde{\mu})}^{C^{\tau-2}S^{-\frac{1}{2}}_{1,\frac{1}{2}}} + \overbrace{\sum_{|\alpha|=2}\frac{1}{\alpha!}D^{\alpha}_{\xi'}b_0\partial^{\alpha}_{x'}(t_1+t_0+\tilde{\mu})}^{C^{\tau-3}S^{-\frac{3}{2}}_{1,\frac{1}{2}}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
Op(t_1+t_0+\tilde{\mu})Op(b_0) - I &= Op(D_{\xi'}(t_1+t_0+\tilde{\mu})\partial_{x'}b_0 \\
&\quad + \sum_{|\alpha|=2}\frac{1}{\alpha!}D^{\alpha}_{\xi'}(t_1+t_0+\tilde{\mu})\partial^{\alpha}_{x'}b_0) + \mathcal{R}^r_0 \\
Op(b_0)Op(t_1+t_0+\tilde{\mu}) - I &= Op(D_{\xi'}b_0\partial_{x'}(t_1+t_0+\tilde{\mu}) \\
&\quad + \sum_{|\alpha|=2}\frac{1}{\alpha!}D^{\alpha}_{\xi'}b_0\partial^{\alpha}_{x'}(t_1+t_0+\tilde{\mu})) + \mathcal{R}^l_0.
\end{aligned}$$

$$\mathcal{R}^r_0 : H^{s+1-\frac{\theta_{12}}{2}}(\partial D) \to H^s(\partial D),$$

is bounded for $\theta_{12} \in (2,3)$, $-(\tau-2) < s < \tau-2$ and $-\frac{1}{2}(\tau-1-\theta_{12}) < s+1 < \tau-1$, and

$$\mathcal{R}^l_0 : H^{s+1-\theta_{12}}(\partial D) \to H^s(\partial D),$$

is bounded for $\theta_{12} \in (2,3)$, $-\frac{1}{2}(\tau-2) < s < \tau-2$ and $-(\tau-1-\theta_{12}) < s < \tau-1$.

$$Op(D_{\xi'}(t_1+t_0+\tilde{\mu})\partial_{x'}b_0 + \sum_{|\alpha|=2}\frac{1}{\alpha!}D^{\alpha}_{\xi'}(t_1+t_0+\tilde{\mu})\partial^{\alpha}_{x'}b_0) : H^{s-\frac{1}{2}}(\partial D) \to H^s(\partial D),$$

and

$$Op(D_{\xi'}b_0\partial_{x'}(t_1+t_0+\tilde{\mu}) + \sum_{|\alpha|=2}\frac{1}{\alpha!}D^{\alpha}_{\xi'}b_0\partial^{\alpha}_{x'}(t_1+t_0+\tilde{\mu})) : H^{s-\frac{1}{2}}(\partial D) \to H^s(\partial D),$$

are bounded if $-\frac{1}{2}(\tau-3) < s < \tau-3$. Therefore $Op(t_1+t_0+\tilde{\mu})$ is a Fredholm operator for $\{|\tilde{\mu}| < c\}$ or $\tilde{\mu}$ outside a suitable sector contained around the positive real axis for $|\xi'| \geq C$ with ind $Op(t_1+t_0+\tilde{\mu}) = 0$ from Corollary (2.5.5) and since we can write

$$T(\tilde{\lambda}) + \tilde{\mu} = \overbrace{Op(t_1+t_0+\tilde{\mu})}^{\text{Fredholm operator}} + \overbrace{Op(t_{-1}+t_{-2}+\cdots)}^{\text{compact operator}}$$

$T(\tilde{\lambda}) + \tilde{\mu}$ is also Fredholm operator with

$$\text{ind } T(\tilde{\lambda}) + \tilde{\mu} = \text{ ind } Op(t_1+t_0+\tilde{\mu}).$$

Hence $T(\lambda)$ is a Fredholm operator with zero index.

For $\psi \in D(T(\lambda))$

$$\|Op(b_0)T(\lambda)\psi\|_{H^{s+\frac{3}{2}}(\partial D)} \leq C\|T(\lambda)\psi\|_{H^{s+\frac{3}{2}}(\partial D)}, \tag{4.29}$$

and if we set

$$\begin{aligned}\overline{\mathcal{R}} = Op(b_0)Op(t_1 + t_0 - \tilde{\mu}) - I &= Op(D_{\xi'} b_0 \partial_{x'}(t_1 + t_0 - \tilde{\mu}) \\ &\quad + \sum_{|\alpha|=2} \frac{1}{\alpha!} D^{\alpha}_{\xi'} b_0 \partial^{\alpha}_{x'}(t_1 + t_0 - \tilde{\mu})) + \mathcal{R}^l_0 \\ &: H^{s-\frac{1}{2}}(\partial D) \to H^s(\partial D)\end{aligned}$$

is bounded if $-\frac{1}{2}(\tau - 3) < s < \tau - 3$, then

$$\|(I + \overline{\mathcal{R}})\psi\|_{H^{s+\frac{3}{2}}(\partial D)} \geq \|\psi\|_{H^{s+\frac{3}{2}}(\partial D)} - \|\overline{\mathcal{R}}\psi\|_{H^s(\partial D)}. \tag{4.30}$$

From (4.29) and (4.30) we conclude the following a priori estimate,

$$\|\psi\|_{H^{s+\frac{3}{2}}(\partial D)} \leq \|T(\lambda)\psi\|_{H^{s+\frac{3}{2}}(\partial D)} + \|\psi\|_{H^{s-\frac{1}{2}}(\partial D)}. \tag{4.31}$$

4.3 A Priori Estimates

In this section we study the operator A_L, and prove an a priori estimate for the operator $A_L - \lambda$ which will play a fundamental role in showing that $T(\lambda)$ is injective.

We associate with the problem (4.2) a linear operator A_L in $H^s(D)$ acting like A on the domain

$$D(A_L) = \left\{u \in H^{s+2}(D); Lu = 0\right\}.$$

We remark that the operator A_L is closed. Let v_j be an arbitrary sequence in $D(A_L)$ such that $v_j \to v$ in $H^{s+2}(D)$ and $Av_j \to g$ in $H^s(D)$. Since $Av_j \to Av = g \in H^s(D)$ and $\lim Lv_j \to Lv = 0 \in H^{s+\frac{1}{2}}(\partial D)$, then $v \in D(A_L)$.

With the help of next theorem, we show that A_L is injective.

Theorem 4.3.1. *Assume that condition* (4.4) *is satisfied. Then for every* $\eta \in (-\pi, \pi)$, *there exists a constant* $R(\eta) > 0$ *depending on* η *such that if* $\lambda = r^2 e^{i\eta}$, $r \geq 0$ *and* $|\lambda| = r^2 \geq R(\eta)$, *we have for all* $u \in H^{s+2}(D)$ *satisfying* $Lu = 0$ *on* ∂D *(i.e.* $u \in D(A_L)$*)*

$$\|u\|^2_{H^{s+2}} + \langle r\rangle^{s+2}\|u\|^2_{L^2} \leq C''(\eta)\Big(\|(A_L - \lambda)u\|^2_{H^{s+2}} + \langle r\rangle^s\|(A_L - \lambda)u\|^2_{L^2}\Big),$$

with a constant $C''(\eta) > 0$, *and for*

$$0 \leq s < \tau - 3.$$

Proof. We introduce an auxiliary variable t in the unit circle $S = \mathbb{R}/2\pi\mathbb{Z}$, and replace the parameter λ by the second-order differential operator $-e^{i\eta}D^2_t$ for $-\pi < \eta < \pi$. We consider instead of the problem

$$\begin{cases} (A_L - \lambda)u = f & \text{in } D \\ Lu = \mu_1(x')\frac{\partial u}{\partial n}(x') + \mu_2(x')u(x') = 0 & \text{on } \partial D \end{cases}$$

the following boundary problem

$$\begin{cases} \Pi(\eta)\tilde{u} = (A_L + e^{i\eta}D_t^2)\tilde{u} = \tilde{f} & \text{in } D \times S \\ L\tilde{u} = \mu_1(x')\frac{\partial \tilde{u}}{\partial n}(x') + \mu_2(x')\tilde{u}(x')|_{\partial D\times S} = 0 & \text{on } \partial D \times S \end{cases}. \tag{4.32}$$

We reduce the study of problem (4.32) to that of a pseudodifferential operator on the boundary. According to the Sections 3.1 and 3.2, there exist $\begin{pmatrix} R^D(\eta) & K^D(\eta) \end{pmatrix}$ and $\begin{pmatrix} R^N(\eta) & K^N(\eta) \end{pmatrix}$ such that

$$\begin{pmatrix} \Pi(\eta) = (A_L + e^{i\eta}D_t^2) \\ \gamma_0 \end{pmatrix} \begin{pmatrix} R^D(\eta) & K^D(\eta) \end{pmatrix} = I,$$

$$\begin{pmatrix} \Pi(\eta) = (A_L + e^{i\eta}D_t^2) \\ \gamma_1 \end{pmatrix} \begin{pmatrix} R^N(\eta) & K^N(\eta) \end{pmatrix} = I.$$

Then we consider instead of (4.6), the following operator

$$\begin{pmatrix} \Pi(\eta) \\ L \end{pmatrix} \begin{pmatrix} R^N(\eta) & K^D(\eta) \end{pmatrix} = \begin{pmatrix} I & 0 \\ LR^N(\eta) & LK^D(\eta) \end{pmatrix} : \begin{matrix} H^s(D \times S) \\ \times \\ D(\tilde{T}(\eta)) \end{matrix} \to \begin{matrix} H^s(D \times S) \\ \times \\ H^{s+\frac{3}{2}}(\partial D \times S) \end{matrix}$$

with

$$\begin{aligned} \tilde{T}(\eta) = LK^D(\eta) : D(\tilde{T}(\eta)) \subset H^{s+\frac{3}{2}}(\partial D \times S) &\to H^{s+\frac{3}{2}}(\partial D \times S) \\ \tilde{\psi} &\longmapsto LK^D(\eta)(\tilde{\psi}), \end{aligned}$$

and $D(\tilde{T}(\eta)) = \left\{ \tilde{\psi} \in H^{s+\frac{3}{2}}(\partial D \times S); \tilde{T}(\eta)\tilde{\psi} \in H^{s+\frac{3}{2}}(\partial D \times S) \right\}$. We can extend the estimate (4.31) on $\partial D \times S$ i.e.

$$\|\tilde{\psi}\|_{H^{s+\frac{3}{2}}(\partial D\times S)} \leq \|\tilde{T}(\eta)\tilde{\psi}\|_{H^{s+\frac{3}{2}}(\partial D\times S)} + \|\tilde{\psi}\|_{H^{s-\frac{1}{2}}(\partial D\times S)}. \tag{4.33}$$

Now we want to show that for all $\tilde{u} \in H^{s+2}(D \times S)$ satisfying $L\tilde{u} = 0$ on $\partial D \times S$ the following estimate follows from (4.33)

$$\|\tilde{u}\|_{H^{s+2}(D\times S)} \leq C'(\eta)\Big(\|\Pi(\eta)\tilde{u}\|_{H^s(D\times S)} + \|\tilde{u}\|_{H^s(D\times S)}\Big). \tag{4.34}$$

Every element $\tilde{u}(x,t) = u(x)e^{irt} \in H^{s+2}(D \times S)$ satisfying $L\tilde{u} = 0$ on $\partial D \times S$ can be written in the following form:

$$\tilde{u}(x,t) = \tilde{v}(x,t) + \tilde{w}(x,t)$$

where $\tilde{v} \in H^{s+2}(D \times S)$ is the solution of

$$\begin{cases} \Pi(\eta)\tilde{v} = f & \text{in } D \\ \gamma_1\tilde{v} = 0 & \text{on } \partial D \end{cases}$$

for all $-\tau + \frac{1}{2} < s < \tau$ and $s > -\frac{1}{2}$ from Section 3.2, then

$$\|\tilde{v}\|_{H^{s+2}(D\times S)} \leq C(\eta)\|\Pi(\eta)\tilde{u}\|_{H^s(D\times S)} \tag{4.35}$$

since $\tilde{w} = \tilde{u} - \tilde{v} \in N(\Pi(\eta))$. The distribution $\tilde{w}$ can be written as $\tilde{w} = K^D(\eta)\tilde{\psi}$, $\tilde{\psi} = \gamma_0\tilde{w} \in H^{s+\frac{3}{2}}(\partial D \times S)$. Applying (4.33) to the $\gamma_0\tilde{w}$ then

$$\begin{aligned}
\|\gamma_0\tilde{w}\|_{H^{s+\frac{3}{2}}(\partial D\times S)} &\leq \|\tilde{T}(\eta)\gamma_0\tilde{w}\|_{H^{s+\frac{3}{2}}(\partial D\times S)} + \|\gamma_0\tilde{w}\|_{H^{s-\frac{1}{2}}(\partial D\times S)} \\
&\leq \|L\tilde{w}\|_{H^{s+\frac{3}{2}}(\partial D\times S)} + \|\gamma_0\tilde{w}\|_{H^{s-\frac{1}{2}}(\partial D\times S)} \\
&\leq \|L\tilde{u}\|_{H^{s+\frac{3}{2}}(\partial D\times S)} + \|L\tilde{v}\|_{H^{s+\frac{3}{2}}(\partial D\times S)} + \|\gamma_0\tilde{u}\|_{H^{s-\frac{1}{2}}(\partial D\times S)} \\
&\quad + \|\gamma_0\tilde{v}\|_{H^{s-\frac{1}{2}}(\partial D\times S)} \\
&\overbrace{\leq}^{(4.35)} C(\eta)\|\Pi(\eta)\tilde{u}\|_{H^s(D\times S)} + \|\tilde{u}\|_{H^s(D\times S)} + \|\tilde{v}\|_{H^s(D\times S)}, \quad (4.36)
\end{aligned}$$

and with (4.36)

$$\begin{aligned}
\|\tilde{w}\|_{H^{s+2}(D\times S)} &= \|K^D(\eta)\gamma_0\tilde{w}\|_{H^{s+2}(D\times S)} \leq \|\tilde{\psi}\|_{H^{s+\frac{3}{2}}(D\times S)} \\
&\leq C(\eta)\|\Pi(\eta)\tilde{u}\|_{H^s(D\times S)} + \|\tilde{u}\|_{H^s(D\times S)} + \|\tilde{v}\|_{H^{s+2}(D\times S)}. \quad (4.37)
\end{aligned}$$

Then from (4.35) and (4.37) , we have

$$\begin{aligned}
\|\tilde{u}\|_{H^{s+2}(D\times S)} &\leq \|\tilde{v}\|_{H^{s+2}(D\times S)} + \|\tilde{w}\|_{H^{s+2}(D\times S)} \\
&\leq C'(\eta)\Big(\|\Pi(\eta)\tilde{u}\|_{H^s(D\times S)} + \|\tilde{u}\|_{H^s(D\times S)}\Big),
\end{aligned}$$

or

$$\|u(x)e^{irt}\|_{H^{s+2}(D\times S)} \leq C'(\eta)\Big(\|\Pi(\eta)u(x)e^{irt}\|_{H^s(D\times S)} + \|u(x)e^{irt}\|_{H^s(D\times S)}\Big). \quad (4.38)$$

We can estimate each term of inequality (4.38) by [8, A.26] as follows:

$$\|u(x)e^{irt}\|_{H^s(D\times S)} \simeq \Big(\|u\|^2_{H^s(D)} + \langle r\rangle^s\|u\|^2_{L^2(D)}\Big)^{\frac{1}{2}}. \quad (4.39)$$

$$\begin{aligned}
\|\Pi(\eta)u(x)e^{irt}\|_{H^s(D\times S)} &= \|(A_L + e^{i\eta}D_t^2)u(x)e^{irt}\|_{H^s(D\times S)} \quad (4.40) \\
&\cong \Big(\|(A_L + e^{i\eta t})u\|^2_{H^s(D)} + \langle r\rangle^s\|(A_L + e^{i\eta t})u\|^2_{L^2(D)}\Big)^{\frac{1}{2}}.
\end{aligned}$$

$$\|u(x)e^{irt}\|_{H^{s+2}(D\times S)} \cong \Big(\|u\|^2_{H^{s+2}(D)} + \langle r\rangle^{s+2}\|u\|^2_{L^2(D)}\Big)^{\frac{1}{2}}. \quad (4.41)$$

If we carry equalities (4.39),(4.40) and (4.41) into inequality (4.38), then we have

$$\begin{aligned}
&\|u\|^2_{H^{s+2}(D)} + \langle r\rangle^{s+2}\|u\|^2_{L^2(D)} \quad (4.42) \\
&\leq C'(\eta)\Big(\|(A_L - r^2e^{i\eta t})u\|^2_{H^s(D)} + \langle r\rangle^s\|(A_L - r^2e^{i\eta t})u\|^2_{L^2(D)} + \|u\|^2_{H^s(D)} + \langle r\rangle^s\|u\|^2_{L^2(D)}\Big).
\end{aligned}$$

To eliminate the term $C'(\eta)\|u\|_{H^s(D)}$ on the right hand side of (4.42), we need the following inequalities [19, Chapter 8, Section 8.4, (17), (18)]:

i) For every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\|v_1\|^2_{H^{s+1}(\overline{D})} \leq \epsilon\|v_1\|^2_{H^{s+2}(\overline{D})} + C_\epsilon\|v_1\|^2_{L^2(D)}, \; v_1 \in H^{s+2}(\overline{D}). \quad (4.43)$$

ii) There exists a constant $C_1 > 0$ independent of $C'(\eta) \geq 0$ such that for $s \geq 0$

$$C'(\eta)\|v_2\|^2_{H^s(\overline{D})} \leq C_1\Big(\|v_2\|^2_{H^{s+1}(\overline{D})} + C'(\eta)^{s+1}\|v_2\|^2_{L^2(D)}\Big)\,,\ v_2 \in H^{s+1}(\overline{D}). \tag{4.44}$$

Applying inequalities (4.43) and (4.44) to the function u and taking $\epsilon = \frac{1}{2C_1C_2}$, we have

$$C'(\eta)C_2\|u\|^2_{H^s(D)} \leq \frac{1}{2}\|u\|^2_{H^{s+2}(D)} + C'(\eta)^{s+1}C''\|u\|^2_{L^2(D)}, \tag{4.45}$$

with a constant $C'' > 0$. Therefore, carrying (4.45) into inequality (4.42), we have

$$\begin{aligned}
&\|u\|^2_{H^{s+2}(D)} + \langle r\rangle^{s+2}\|u\|^2_{L^2(D)} \\
&\quad\leq C'(\eta)\Big(\|(A_L - r^2e^{i\eta t})u\|^2_{H^s(D)} + \langle r\rangle^s\|(A_L - r^2e^{i\eta t})u\|^2_{L^2(D)} \\
&\quad+ \frac{1}{2C_2}\|u\|^2_{H^{s+2}(D)} + C'(\eta)^s\frac{C''}{C_2}\|u\|^2_{L^2(D)} + \langle r\rangle^s\|u\|^2_{L^2(D)}\Big),
\end{aligned}$$

then with another constant $C_3 > 0$,

$$\begin{aligned}
&\|u\|^2_{H^{s+2}(D)} + \langle r\rangle^{s+2}\|u\|^2_{L^2(D)} \\
&\quad\leq C'(\eta)C_3\Big(\|(A_L - r^2e^{i\eta t})u\|^2_{H^s(D)} + \langle r\rangle^s\|(A_L - r^2e^{i\eta t})u\|^2_{L^2(D)} + C'(\eta)^s\langle r\rangle^s\|u\|^2_{L^2(D)}\Big).
\end{aligned}$$

If r is so large that

$$r \geq 2C'(\eta)C_3,$$

then we can eliminate the last term $C'(\eta)^s\langle r\rangle^s\|u\|^2_{L^2(D)}$ on the right hand side and we have

$$\begin{aligned}
&\|u\|^2_{H^{s+2}(D)} + \langle r\rangle^{s+2}\|u\|^2_{L^2(D)} \\
&\quad\leq 2C'(\eta)C_3\Big(\|(A_L - r^2e^{i\eta t})u\|^2_{H^s(D)} + \langle r\rangle^s\|(A_L - r^2e^{i\eta t})u\|^2_{L^2(D)}\Big).
\end{aligned} \tag{4.46}$$

If we take $C''(\eta) = 2C'(\eta)C_3$, $\lambda = r^2e^{i\eta t}$ and $R(\eta) = 4C'^2(\eta)C_3^2$, then the proof is complete. ◻

4.4 Proof of Theorem 4.0.4

In this section we prove Theorem 4.0.4.

Proof. We remark that

$$N\big((A-\lambda, L)\big) = \big\{u \in H^{s+2}_L; (A-\lambda, L)u = 0\big\} = N(A_L - \lambda).$$

From (4.46) $A_L - \lambda$ is injective, therefore $(A-\lambda, L)$ is injective. Now we show that if $(A-\lambda, L)$ is injective, then $T(\lambda)$ is injective.
Let $\psi \in H^{s+\frac{3}{2}}$ and $T(\lambda)\psi = 0$. We set $K^D(\lambda)\psi = u$. Then

$$\begin{pmatrix} A-\lambda \\ L \end{pmatrix} u = \begin{pmatrix} (A-\lambda)K^D(\lambda)\psi \\ LK^D(\lambda)\psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $u \in N\big((A-\lambda, L)\big)$ and $\psi = 0$. This shows that $N\big(T(\lambda)\big) = 0$. Since $\operatorname{ind} T(\lambda) = 0$, we conclude that $T(\lambda)$ is also injective. Hence $T(\lambda)$ is invertible and the proof of Theorem 4.0.4 is complete. ◻

Chapter 5

Appendix

5.1 Details on the Resolvent Construction for the Dirichlet Problem

Proof. Consider $\lambda_0 = r^2 e^{i\eta}$, $\eta \in (-\pi, \pi)$ on a ray outside $\overline{\mathbb{R}}_-$. We introduce an extra variable $t \in S^1$ (S^1 is the unit circle), and replace r by $D_t = -i\partial_t$, now consider

$$\tilde{A}^D = A - e^{i\eta} D_t^2 \text{ on } \mathbb{R}^n_+ \times S^1.$$

Since $a(x,\xi) - \lambda_0 \neq 0$, $\tilde{A}^D$ is elliptic on $\mathbb{R}^+ \times S^1$ and $\tilde{\mathcal{A}}^D = \begin{pmatrix} \tilde{A}^D \\ \gamma_0 \end{pmatrix}$ has a parametrix $\tilde{\mathcal{B}}^D_0$ and the remainder $\tilde{\mathcal{R}}^D = \tilde{\mathcal{A}}^D \tilde{\mathcal{B}}^D_0 - I$ as in (3.5), (3.6) with $\mathbb{R}^n_+$, $\mathbb{R}^{n-1}$ replaced by $\mathbb{R}^n_+ \times S^1$, $\mathbb{R}^{n-1} \times S^1$. For functions w of the form $w(x,t) = u(x)e^{ir_0 t}$, $u \in S(\mathbb{R}^n_+)$ and $r_0 \in 2\pi\mathbb{Z}$, we have

$$\tilde{\mathcal{A}}^D = \begin{pmatrix} (A - e^{i\eta} r_0^2) w \\ \gamma_0 w \end{pmatrix} = \begin{pmatrix} (A - \lambda_0) w \\ \gamma_0 w \end{pmatrix},$$

$$\|w\|_{H^s(\mathbb{R}^n_+ \times S^1)} \simeq \|(1 + |\xi|^2 + r_0^2)^{\frac{s}{2}} \hat{u}(\xi)\|_{L_2} = \|u\|_{H^{s,r_0}}.$$

When $s' < s$, then

$$\begin{aligned} \|w\|_{H^{s'}(\mathbb{R}^n_+ \times S^1)} &\simeq \|(1 + |\xi|^2 + r_0^2)^{\frac{s'}{2}} \hat{u}(\xi)\|_{L_2} \\ &\leq \langle r_0 \rangle^{s'-s} \|(1 + |\xi|^2 + r_0^2)^{\frac{s}{2}} \hat{u}(\xi)\|_{L_2} \simeq \langle r_0 \rangle^{s'-s} \|w\|_{H^s(R^n_+ \times S^1)}. \end{aligned}$$

Since the remainder $\tilde{\mathcal{R}}^D$ acts like $\mathcal{R}^D(\lambda_0)$, we find that

$$\begin{aligned} \|\mathcal{R}^D(\lambda_0)\{u_1, u_2\}\|_{H^{s,r_0}(\mathbb{R}^n_+) \times H^{s+\frac{3}{2},r_0}(\mathbb{R}^{n-1})} &\leq c_s \|\{u_1, u_2\}\|_{H^{s-\theta,r_0}(\mathbb{R}^n_+) \times H^{s+\frac{3}{2}-\theta,r_0}(\mathbb{R}^{n-1})} \\ &\leq c'_s \langle r_0 \rangle^{-\theta} \|\{u_1, u_2\}\|_{H^{s,r_0}(\mathbb{R}^n_+) \times H^{s+\frac{3}{2},r_0}(\mathbb{R}^{n-1})} \end{aligned} \tag{5.1}$$

for s as in (3.4). Now we want to extend it to arbitrary λ as follows: Write $\lambda = r^2 e^{i\eta} = (r_0 + r')^2 e^{i\eta}$ with $r' \in [0, 2\pi)$. Since

$$(1 + |\xi|^2 + r_0^2)^{\frac{t}{2}} \simeq (1 + |\xi|^2 + (r_0 + r')^2)^{\frac{t}{2}}, \tag{5.2}$$

$$A - \lambda = A - \lambda_0 + (\lambda_0 - \lambda),$$

$$|\lambda_0 - \lambda| = |r_0^2 - r^2| = |2r_0 r' + r'^2| \leq c\langle r_0\rangle, \tag{5.3}$$

$$\|\mathcal{B}_0^D(\lambda_0)\{u_1,u_2\}\|_{H^{s+2,r}(\mathbb{R}_+)} \leq c_s\|\{u_1,u_2\}\|_{H^{s,r}(\mathbb{R}_+)\times H^{s+\frac{3}{2},r}(\mathbb{R}^{n-1})}, \tag{5.4}$$

$$\begin{aligned}
\|\mathcal{B}_0^D(\lambda_0)\{u_1,u_2\}\|_{H^{s+2,r}(\mathbb{R}_+)} &\simeq \|(1+|\xi|^2+|r_0|^2)^{\frac{s+2}{2}}\mathcal{B}_0^D(\lambda_0)\{u_1,u_2\}\|_{L_2} \\
&\geq \langle r_0\rangle\|(1+|\xi|^2+|r_0|)^{\frac{s+2}{2}}\mathcal{B}_0^D(\lambda_0)\{u_1,u_2\}\|_{L_2} \\
&\geq \langle r_0\rangle\|(1+|\xi|^2+|r_0|)^{\frac{s}{2}}\mathcal{B}_0^D(\lambda_0)\{u_1,u_2\}\|_{L_2} \\
&= \langle r_0\rangle\|\mathcal{B}_0^D(\lambda_0)\{u_1,u_2\}\|_{H^{s,r}(\mathbb{R}_+)},
\end{aligned} \tag{5.5}$$

we have

$$\mathcal{R}^D(\lambda) = \mathcal{A}^D(\lambda)\mathcal{B}_0^D(\lambda_0) - I = \mathcal{A}^D(\lambda_0)\mathcal{B}_0^D(\lambda_0) - I + \begin{pmatrix}\lambda_0-\lambda\\0\end{pmatrix}\mathcal{B}_0^D(\lambda_0). \tag{5.6}$$

Therefore

$$\begin{aligned}
\|\mathcal{R}^D(\lambda)\{u_1,u_2\}\|_{H^{s,r}(\mathbb{R}^n_+)\times H^{s+\frac{3}{2},r}(\mathbb{R}^{n-1})} &\overbrace{\simeq}^{(5.2)} \|\mathcal{R}^D(\lambda)\{u_1,u_2\}\|_{H^{s,r_0}(\mathbb{R}^n_+)\times H^{s+\frac{3}{2},r_0}(\mathbb{R}^{n-1})} \\
&\overbrace{\leq}^{(5.6)} \|\mathcal{R}^D(\lambda_0)\{u_1,u_2\}\|_{H^{s,r_0}(\mathbb{R}^n_+)\times H^{s+\frac{3}{2},r_0}(\mathbb{R}^{n-1})} \\
&\quad +|\lambda_0-\lambda|\ \|\mathcal{B}_0^D(\lambda_0)\{u_1,u_2\}\|_{H^{s,r}(\mathbb{R}_+)} \\
&\overbrace{\leq}^{(5.1),(5.5),(5.4)} d_s\|\{u_1,u_2\}\|_{H^{s,r_0}(\mathbb{R}^n_+)\times H^{s+\frac{3}{2},r_0}(\mathbb{R}^{n-1})}
\end{aligned}$$

If we define $B_0^D(\lambda) = B_0^D(\lambda_0)$, then (5.1) holds for general λ.
Fix s. Consider $\lambda = r^2 e^{i\eta}$ with $r_0 \geq r_1$ for large r_1, where for each s, $c'_s\langle r_0\rangle^{-\theta} \leq \frac{1}{2}$. Then $\|\mathcal{R}^D(\lambda)\|_{H^{s,r}\times H^{s+\frac{3}{2},r}} < 1$ and $I + \mathcal{R}^D(\lambda)$ has the inverse $I + \mathcal{R}''^D(\lambda) = I + \sum_{k\geq 1}(-\mathcal{R}^D(\lambda))^k$ (converging in the operator norm for operators on $H^{s,r}(\mathbb{R}^n_+) \times H^{s+\frac{3}{2},r}(\mathbb{R}^{n-1})$). Therefore by definition of $\mathcal{B}_0^D(\lambda)$,

$$\mathcal{A}^D(\lambda)\mathcal{B}_0^D(\lambda)(I + \mathcal{R}''^D(\lambda)) = I.$$

This gives the right inverse

$$\begin{aligned}
\mathcal{B}^D(\lambda) &= \mathcal{B}_0^D(\lambda) + \mathcal{B}_0^D(\lambda)\mathcal{R}''^D(\lambda) \\
&= \begin{pmatrix}R_0^D(\lambda) & K_0^D(\lambda)\end{pmatrix} + \begin{pmatrix}R_0^D(\lambda) & K_0^D(\lambda)\end{pmatrix}\mathcal{R}''^D(\lambda) = \begin{pmatrix}R^D(\lambda) & K^D(\lambda)\end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{B}_0^D(\lambda)\mathcal{R}''^D(\lambda)\{f,g\}\|_{H^{s+2,r}(\mathbb{R}^n_+)} &\leq c_s\|\{f,g\}\|_{H^{s-\theta,r}(\mathbb{R}^n_+)\times H^{s+\frac{3}{2}-\theta,r}(\mathbb{R}^{n-1})} \\
&\leq c'_s\langle r\rangle^{-\theta}\|\{f,g\}\|_{H^{s,r}(\mathbb{R}^n_+)\times H^{s+\frac{3}{2},r}(\mathbb{R}^{n-1})}.
\end{aligned}$$

Since

$$\mathcal{A}^D(\lambda)\tilde{\mathcal{B}}^D(\lambda) = \begin{pmatrix}(A-\lambda)R^D(\lambda) & (A-\lambda)K^D(\lambda)\\ \gamma_0 R^D(\lambda) & \gamma_0 K^D(\lambda)\end{pmatrix} = \begin{pmatrix}I & 0\\0 & I\end{pmatrix},$$

$R^D(\lambda)$ solves

$$(A-\lambda)u = f\ ,\ \gamma_0 u = 0, \tag{5.7}$$

and $K^D(\lambda)$ solves

$$(A-\lambda)u = 0\ ,\ \gamma_0 u = \varphi. \tag{5.8}$$

For such large λ, $R^D(\lambda)$ coincides with the resolvent $(A^D_{\gamma_0}-\lambda)^{-1}$ of $A^D_{\gamma_0}$. The operator $K^D(\lambda)$ is the Poisson operator, which solves (5.8). Since $\lambda \in \rho(A^D_{\gamma_0})$, it is denoted by $K^\lambda_{\gamma_0}$. For each $\lambda = r^2 e^{i\eta}$, $r_1 \le r$,

$$(A^D_{\gamma_0}-\lambda)^{-1} : H^s(\mathbb{R}^n_+) \to H^{s+2}(\mathbb{R}^n_+)\ ,\ K^\lambda_{\gamma_0} : H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}) \to H^{s+2}(\mathbb{R}^n_+)$$

for s satisfying (3.3).

From above

$$R^D(\lambda) = R^D_0(\lambda) + R^D_0(\lambda)\mathcal{R}''^D(\lambda)\ ,\ K^D(\lambda) = K^\lambda_{\gamma_0} = K^D_0(\lambda) + K^D_0(\lambda)\mathcal{R}''^D(\lambda), \tag{5.9}$$

where

$$\begin{array}{rcl} \| R^D_0(\lambda) \|_{\mathcal{L}(H^{s,r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))} & , & \| K^D_0(\lambda) \|_{\mathcal{L}(H^{s+\frac{3}{2},r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))} \text{ are } O(1) \\ \| R^D_0(\lambda)\mathcal{R}''^D(\lambda) \|_{\mathcal{L}(H^{s-\theta,r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))} & , & \| K^D_0(\lambda)\mathcal{R}''^D(\lambda) \|_{\mathcal{L}(H^{s+\frac{3}{2}-\theta,r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))} \text{ are } O(1) \\ \| R^D_0(\lambda)\mathcal{R}''^D(\lambda) \|_{\mathcal{L}(H^{s,r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))} & , & \| K^D_0(\lambda)\mathcal{R}''^D(\lambda) \|_{\mathcal{L}(H^{s+\frac{3}{2},r}(\mathbb{R}^n_+),H^{s+2,r}(\mathbb{R}^n_+))} \text{ are } O(\langle\lambda\rangle^{-\frac{\theta}{2}}) \end{array} \tag{5.10}$$

for λ on the rays $\lambda = r^2 e^{i\eta}$ as in the statement of the theorem and s as in (3.3) and (3.4). In view of [8, A.26] we have

$$\begin{aligned} \|R^D(\lambda)u_1\|_{H^{s+2,r}} &\cong \Big(\langle\lambda\rangle^{s+2}\|R^D(\lambda)u_1\|_0^2 + \|R^D(\lambda)u_1\|^2_{H^{s+2}}\Big)^{\frac{1}{2}} \\ &\cong C'\langle\lambda\rangle^{\frac{s}{2}+1}\|R^D(\lambda)u_1\|_0^2 + C'\|R^D(\lambda)u_1\|^2_{H^{s+2}} \\ &\le \|u_1\|_{H^{s,r}} \cong C_s\Big(\langle\lambda\rangle^{s}\|u_1\|_0^2 + \|u_1\|^2_{H^s}\Big)^{\frac{1}{2}} \\ &\le C'_s\langle\lambda\rangle^{\frac{s}{2}}\|u_1\|_{H^s} \end{aligned} \tag{5.11}$$

$$\begin{aligned} \|K^D(\lambda)u_2\|_{H^{s+2,r}} &\cong \Big(\langle\lambda\rangle^{s+2}\|K^D(\lambda)u_2\|_0^2 + \|K^D(\lambda)u_2\|^2_{H^{s+2}}\Big)^{\frac{1}{2}} \\ &\cong C'\langle\lambda\rangle^{\frac{s}{2}+1}\|K^D(\lambda)u_2\|_0^2 + C'\|K^D(\lambda)u_2\|^2_{H^{s+2}} \\ &\le \|u_2\|_{H^{s+\frac{3}{2},r}} \cong C_s\Big(\langle\lambda\rangle^{s+\frac{3}{2}}\|u_2\|_0^2 + \|u_2\|^2_{s+\frac{3}{2}}\Big)^{\frac{1}{2}}. \end{aligned} \tag{5.12}$$

Note that $\|R^D(\lambda)\|_{\mathcal{L}(L^2(\mathbb{R}^n_+))}$ is $O(\langle\lambda\rangle^{-1})$ on the ray. □

5.2 Details on the Calculation of the Second Term in the Asymptotic Expansion of the Symbol $C^+_{\lambda,00}$ in the Calderón Projector

$$(r_2)_{-2}(x',\xi',\lambda) = \frac{1}{2\pi}\int_{\Gamma_{\xi'}} r_{-3}(x',0,\xi',\xi_n,\lambda)d\xi_n = -\frac{1}{2\pi}\sum_{j=1}^{n-1}\Big(\int_{\Gamma_{\xi'}} \frac{\partial_{x_j}A_2(x')D_{\xi_j}(\kappa_1-\xi_n)}{A_2^2(x')(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)}d\xi_n$$

$$+\int_{\Gamma_{\xi'}} \frac{\partial_{x_j}A_2(x')D_{\xi_j}(\kappa_2-\xi_n)}{A_2^2(x')(\kappa_1-\xi_n)(\kappa_2-\xi_n)^2}d\xi_n + \int_{\Gamma_{\xi'}} \frac{D_{\xi_j}(\kappa_1-\xi_n)\partial_{x_j}(\kappa_1-\xi_n)}{A_2(x')(\kappa_1-\xi_n)^3(\kappa_2-\xi_n)}d\xi_n$$

$$+\int_{\Gamma_{\xi'}} \frac{D_{\xi_j}(\kappa_2-\xi_n)\partial_{x_j}(\kappa_1-\xi_n)}{A_2(x')(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)^2}d\xi_n + \int_{\Gamma_{\xi'}} \frac{D_{\xi_j}(\kappa_1-\xi_n)\partial_{x_j}(\kappa_2-\xi_n)}{A_2(x')(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)^2}d\xi_n$$

$$+\int_{\Gamma_{\xi'}} \frac{D_{\xi_j}(\kappa_2-\xi_n)\partial_{x_j}(\kappa_2-\xi_n)}{A_2(x')(\kappa_1-\xi_n)(\kappa_2-\xi_n)^3}d\xi_n\Big) - \frac{1}{2\pi}\int_{\Gamma_{\xi'}} \frac{ia_1(x',\xi')}{A_2^2(x')(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)^2}d\xi_n$$

$$= \quad -\frac{\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_1}{2\pi A_2^2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^2}d\xi_n - \frac{i\partial_{x_n}A_2(x')}{2\pi A_2^2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^2}d\xi_n$$

$$-\frac{\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_2}{2\pi A_2^2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{\kappa_1-\xi_n}d\xi_n - \frac{i\partial_{x_n}A_2(x')}{2\pi A_2^2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{\kappa_1-\xi_n}d\xi_n$$

$$-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_1}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^3}d\xi_n - \frac{i(\partial_{x_n}\kappa_1)(x')}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^3}d\xi_n$$

$$-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_1}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}d\xi_n - \frac{i(\partial_{x_n}\kappa_1)(x')}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}d\xi_n$$

$$-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_2}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}d\xi_n - \frac{i(\partial_{x_n}\kappa_2)(x')}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}d\xi_n$$

$$-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_2}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{(\kappa_2-\xi_n)^3}}{\kappa_1-\xi_n}d\xi_n - \frac{i(\partial_{x_n}\kappa_2)(x')}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{(\kappa_2-\xi_n)^3}}{\kappa_1-\xi_n}d\xi_n$$

$$-\frac{ia_1(x',\xi')}{2\pi A_2^2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}d\xi_n$$

$$= \quad -\frac{\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_1}{2\pi A_2^2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^2} - \frac{i\partial_{x_n}A_2(x')}{2\pi A_2^2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^2}$$

$$
\begin{aligned}
&-\frac{\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_2}{2\pi A_2^2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{\kappa_1-\xi_n}-\frac{i\partial_{x_n}A_2(x')}{2\pi A_2^2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{\kappa_1-\xi_n}\\
&-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_1}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^3}-\frac{i(\partial_{x_n}\kappa_1)(x')}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^3}\\
&-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_1}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}-\frac{i(\partial_{x_n}\kappa_1)(x')}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}\\
&-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_2}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}-\frac{i(\partial_{x_n}\kappa_2)(x')}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}\\
&-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_2}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{(\kappa_2-\xi_n)^3}}{\kappa_1-\xi_n}-\frac{i(\partial_{x_n}\kappa_2)(x')}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{(\kappa_2-\xi_n)^3}}{\kappa_1-\xi_n}\\
&-\frac{ia_1(x',\xi')}{2\pi A_2^2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{1}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}\\
=\;&-\frac{i\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_1}{A_2^2(x')}\frac{1}{(\kappa_2-\kappa_1)^2}+\frac{\partial_{x_n}A_2(x')}{A_2^2(x')}\frac{1}{(\kappa_2-\kappa_1)^2}\\
&-\frac{i\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_2}{A_2^2(x')}\frac{1}{(\kappa_2-\kappa_1)^2}+\frac{\partial_{x_n}A_2(x')}{A_2^2(x')}\frac{1}{(\kappa_2-\kappa_1)^2}\\
&-\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_1}{A_2(x')}\frac{2}{(\kappa_2-\kappa_1)^3}+\frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')}\frac{2}{(\kappa_2-\kappa_1)^3}\\
&-\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_1}{A_2(x')}\frac{2}{(\kappa_2-\kappa_1)^3}+\frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')}\frac{2}{(\kappa_2-\kappa_1)^3}\\
&-\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_2}{A_2(x')}\frac{2}{(\kappa_2-\kappa_1)^3}+\frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')}\frac{2}{(\kappa_2-\kappa_1)^3}\\
&-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_2}{A_2(x')}\frac{1}{(\kappa_2-\kappa_1)^3}+\frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')}\frac{1}{(\kappa_2-\kappa_1)^3}+\frac{a_1(x',\xi')}{A_2^2(x')}\frac{2}{(\kappa_2-\kappa_1)^3}\\
=\;&\frac{1}{(\kappa_1-\kappa_2)^2}\Bigg[\frac{-i\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_1}{A_2^2(x')}+\frac{\partial_{x_n}A_2(x')}{A_2^2(x')}
\end{aligned}
$$

$$
-\frac{i\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_2}{A_2^2(x')}+\frac{\partial_{x_n}A_2(x')}{A_2^2(x')}+\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_1}{A_2(x')}\frac{2}{\kappa_1-\kappa_2}-\frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')}\frac{2}{\kappa_1-\kappa_2}
$$

$$
+\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_1}{A_2(x')}\frac{2}{\kappa_1-\kappa_1}-\frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')}\frac{2}{\kappa_1-\kappa_2}+\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_2}{A_2(x')}\frac{2}{\kappa_1-\kappa_2}
$$

$$
+\frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')}\frac{2}{\kappa_1-\kappa_2}+\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_2}{A_2(x')}\frac{1}{\kappa_1-\kappa_2}-\frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')}\frac{1}{\kappa_1-\kappa_2}
$$

$$
+\frac{a_1(x',\xi')}{A_2^2(x')}\frac{2}{\kappa_1-\kappa_2}\Bigg]=\frac{b_1}{(\kappa_1-\kappa_2)^2}
$$

$$
(r_4)_{-1}(x',\xi',\lambda)=\frac{1}{2\pi}\int_{\Gamma_{\xi'}}r_{-3}(x',0,\xi',\xi_n,\lambda)\xi_n d\xi_n=-\frac{1}{2\pi}\sum_{j=1}^{n}\Bigg(\int_{\Gamma_{\xi'}}\frac{\partial_{x_j}A_2(x')D_{\xi_j}(\kappa_1-\xi_n)\xi_n}{A_2^2(x')(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)}d\xi_n
$$

$$
+\int_{\Gamma_{\xi'}}\frac{\partial_{x_j}A_2(x')D_{\xi_j}(\kappa_2-\xi_n)\xi_n}{A_2^2(x')(\kappa_1-\xi_n)(\kappa_2-\xi_n)^2}d\xi_n+\int_{\Gamma_{\xi'}}\frac{D_{\xi_j}(\kappa_1-\xi_n)\partial_{x_j}(\kappa_1-\xi_n)\xi_n}{A_2(x')(\kappa_1-\xi_n)^3(\kappa_2-\xi_n)}d\xi_n
$$

$$
+\int_{\Gamma_{\xi'}}\frac{D_{\xi_j}(\kappa_2-\xi_n)\partial_{x_j}(\kappa_1-\xi_n)\xi_n}{A_2(x')(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)^2}d\xi_n+\int_{\Gamma_{\xi'}}\frac{D_{\xi_j}(\kappa_1-\xi_n)\partial_{x_j}(\kappa_2-\xi_n)\xi_n}{A_2(x')(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)^2}d\xi_n
$$

$$
+\int_{\Gamma_{\xi'}}\frac{D_{\xi_j}(\kappa_2-\xi_n)\partial_{x_j}(\kappa_2-\xi_n)\xi_n}{A_2(x')(\kappa_1-\xi_n)(\kappa_2-\xi_n)^3}d\xi_n\Bigg)-\frac{1}{2\pi}\int_{\Gamma_{\xi'}}\frac{ia_1(x',\xi')\xi_n}{A_2^2(x')(\kappa_1-\xi_n)^2(\kappa_2-\xi_n)^2}d\xi_n
$$

$$
=\quad-\frac{\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_1}{2\pi A_2^2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^2}d\xi_n-\frac{i\partial_{x_n}A_2(x')}{2\pi A_2^2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^2}d\xi_n
$$

$$
-\frac{\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_2}{2\pi A_2^2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{\kappa_1-\xi_n}d\xi_n-\frac{i\partial_{x_n}A_2(x')}{2\pi A_2^2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{\kappa_1-\xi_n}d\xi_n
$$

$$
-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_1}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^3}d\xi_n-\frac{i(\partial_{x_n}\kappa_1)(x')}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^3}d\xi_n
$$

$$
-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_1}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}d\xi_n-\frac{i(\partial_{x_n}\kappa_1)(x')}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}d\xi_n
$$

$$
-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_2}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}d\xi_n-\frac{i(\partial_{x_n}\kappa_2)(x')}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}d\xi_n
$$

$$
-\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_2}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^3}}{\kappa_1-\xi_n}d\xi_n-\frac{i(\partial_{x_n}\kappa_2)(x')}{2\pi A_2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^3}}{\kappa_1-\xi_n}d\xi_n
$$

$$
\begin{aligned}
& -\frac{ia_1(x',\xi')}{2\pi A_2^2(x')}\int_{\Gamma_{\xi'}}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}d\xi_n \\
= \; & -\frac{\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_1}{2\pi A_2^2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^2}-\frac{i\partial_{x_n}A_2(x')}{2\pi A_2^2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^2} \\
& -\frac{\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_2}{2\pi A_2^2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{\kappa_1-\xi_n}-\frac{i\partial_{x_n}A_2(x')}{2\pi A_2^2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{\kappa_1-\xi_n} \\
& -\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_1}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^3}-\frac{i(\partial_{x_n}\kappa_1)(x')}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{\kappa_2-\xi_n}}{(\kappa_1-\xi_n)^3} \\
& -\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_1}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}-\frac{i(\partial_{x_n}\kappa_1)(x')}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2} \\
& -\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_2}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2}-\frac{i(\partial_{x_n}\kappa_2)(x')}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2} \\
& -\frac{\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_2}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^3}}{\kappa_1-\xi_n}-\frac{i(\partial_{x_n}\kappa_2)(x')}{2\pi A_2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^3}}{\kappa_1-\xi_n} \\
& -\frac{ia_1(x',\xi')}{2\pi A_2^2(x')}\mathrm{Res}_{\xi_n=\kappa_1}\frac{\frac{\xi_n}{(\kappa_2-\xi_n)^2}}{(\kappa_1-\xi_n)^2} \\
= \; & -\frac{i\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_1}{A_2^2(x')}\frac{\kappa_2}{(\kappa_2-\kappa_1)^2}+\frac{\partial_{x_n}A_2(x')}{A_2^2(x')}\frac{\kappa_2}{(\kappa_2-\kappa_1)^2} \\
& -\frac{i\sum_{j=1}^{n-1}\partial_{x_j}A_2(x')D_{\xi_j}\kappa_2}{A_2^2(x')}\frac{\kappa_1}{(\kappa_2-\kappa_1)^2}+\frac{\partial_{x_n}A_2(x')}{A_2^2(x')}\frac{\kappa_1}{(\kappa_2-\kappa_1)^2} \\
& -\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_1}{A_2(x')}\frac{2\kappa_2}{(\kappa_2-\kappa_1)^3}+\frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')}\frac{2\kappa_2}{(\kappa_2-\kappa_1)^3} \\
& -\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_2\partial_{x_j}\kappa_1}{A_2(x')}\frac{\kappa_2+\kappa_1}{(\kappa_2-\kappa_1)^3}+\frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')}\frac{\kappa_2+\kappa_1}{(\kappa_2-\kappa_1)^3} \\
& -\frac{i\sum_{j=1}^{n-1}D_{\xi_j}\kappa_1\partial_{x_j}\kappa_2}{A_2(x')}\frac{\kappa_2+\kappa_1}{(\kappa_2-\kappa_1)^3}+\frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')}\frac{\kappa_2+\kappa_1}{(\kappa_2-\kappa_1)^3}
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{\sum_{j=1}^{n-1} D_{\xi_j}\kappa_2 \partial_{x_j}\kappa_2}{A_2(x')} \frac{\kappa_1}{(\kappa_2-\kappa_1)^3} + \frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')} \frac{\kappa_1}{(\kappa_2-\kappa_1)^3} + \frac{a_1(x',\xi')}{A_2^2(x')} \frac{\kappa_2+\kappa_1}{(\kappa_2-\kappa_1)^3} \\
= \; & \frac{\kappa_1+\kappa_2}{(\kappa_1-\kappa_2)^2} \Big[\frac{-i\sum_{j=1}^{n-1} \partial_{x_j} A_2(x') D_{\xi_j}\kappa_1}{A_2^2(x')} \frac{\kappa_2}{\kappa_1+\kappa_2} + \frac{\partial_{x_n} A_2(x')}{A_2^2(x')} \frac{\kappa_2}{\kappa_1+\kappa_2} \\
& - \frac{i\sum_{j=1}^{n-1} \partial_{x_j} A_2(x') D_{\xi_j}\kappa_2}{A_2^2(x')} \frac{\kappa_1}{\kappa_1+\kappa_2} + \frac{\partial_{x_n} A_2(x')}{A_2^2(x')} \frac{\kappa_1}{\kappa_1+\kappa_2} \\
& + \frac{i\sum_{j=1}^{n-1} D_{\xi_j}\kappa_1 \partial_{x_j}\kappa_1}{A_2(x')} \frac{2\kappa_2}{\kappa_1^2-\kappa_2^2} - \frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')} \frac{2\kappa_2}{\kappa_1^2-\kappa_2^2} \\
& + \frac{i\sum_{j=1}^{n-1} D_{\xi_j}\kappa_2 \partial_{x_j}\kappa_1}{A_2(x')} \frac{1}{\kappa_1-\kappa_2} - \frac{(\partial_{x_n}\kappa_1)(x')}{A_2(x')} \frac{1}{\kappa_1-\kappa_2} \\
& + \frac{i\sum_{j=1}^{n-1} D_{\xi_j}\kappa_1 \partial_{x_j}\kappa_2}{A_2(x')} \frac{1}{\kappa_1-\kappa_2} + \frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')} \frac{1}{\kappa_1-\kappa_2} \\
& - \frac{\sum_{j=1}^{n-1} D_{\xi_j}\kappa_2 \partial_{x_j}\kappa_2}{A_2(x')} \frac{\kappa_1}{\kappa_1^2-\kappa_2^2} + \frac{(\partial_{x_n}\kappa_2)(x')}{A_2(x')} \frac{\kappa_1}{\kappa_1^2-\kappa_2^2} + \frac{a_1(x',\xi')}{A_2^2(x')} \frac{1}{\kappa_1^2-\kappa_2^2} \Big] \\
= \; & \frac{(\kappa_1+\kappa_2) b_2}{(\kappa_1-\kappa_2)^2},
\end{aligned}
$$

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