Semistable Reduction of Prime-Cyclic Galois Covers

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Abstract

In our recent article with Wewers [AW11], we have presented a new proof of the well-known Semistable Reduction Theorem of Deligne and Mumford [DM69], which states that all smooth projective and absolutely irreducible curves over a local field have a semistable model, provided one allows to replace the base field by a finite extension. Though already the first proof of Deligne and Mumford is constructive in a theoretical sense, it is in practice very hard to find these specific models or even a suitable field extension as in the statement of the theorem. The goal of [AW11] was to provide an alternative proof that allows to come up with a practical algorithm to actually compute semistable reductions of curves, at least in important special cases.

Our approach to finding a semistable model for a given curve is based on the following observation: when the curve is considered as a non-constant separable cover of the projective line and if the base field is large enough, the Semistable Reduction Theorem guarantees the existence of a semistable model for the projective line whose normalization is a semistable model for the curve under consideration. Without relying on this fact, we take the reversed approach and construct both a large enough extension of the base field and a suitable model for the projective line that induces a semistable model for the initial curve. The rational components of these projective line models can be interpreted as inducing modifications of the original curve model; our task was thus to find modifications that improve the singularities of the curve model. The crucial point is that these modifications can be derived from purely local considerations by examining the singularities one by one. With methods from rigid analytic and formal geometry, the Semistable Reduction Theorem can then be reduced to a more local statement on coverings of the rigid analytic open unit disk. With theoretical arguments, it is possible to further reduce to two base cases; namely, it suffices to determine the improving modifications for prime-cyclic coverings of the open unit disk and for prime-cyclic coverings of open annuli.

In both of these cases, the improving modifications can be constructed explicitly—and this is precisely the purpose of the present dissertation. The way to find these modifications is as follows: A covering of the considered form with prime degree p is given by a Kummer type equation $y^p = f$, with f a power series or a Laurent series. The crucial step is to carefully approximate f by a pth power h^p 'as good as possible'; then the modification searched for can be deduced from the Newton polygon of $f - h^p$. The main tool to obtain these so-called sufficiently precise approximations is by extending the method of p-Taylor expansion introduced by Matignon [Mat03] in his study of the equidistant subcase. One of the reasons why formal p-Taylor expansions prove so useful is that a field extension as in the statement of the Semistable Reduction Theorem is automatically produced. The result is that we can explicitly calculate semistable models for arbitrary cyclic coverings of prime degree. Note that this also completes the new proof of the Semistable Reduction Theorem started in [AW11].

In addition to the above, we also show how to use our results on prime-cyclic covers to handle an arbitrary cyclic covering of prime power degree. Moreover, we demonstrate all our methods by means of suitable examples. For some of the more complicated calculations, we have used the computer algebra system MAGMA [Magma]; the source code can be found at the respective places in the text.

Keywords: Algorithmic Semistable Reduction, Prime-Cyclic Galois Covers, Formal p-Taylor Expansion

Kurzzusammenfassung

In einer unserer jüngeren Arbeiten mit Wewers [AW11] haben wir einen neuen Beweis des wohlbekannten Theorems von Deligne und Mumford [DM69] über die semistabile Reduktion von Kurven dargelegt. Die Aussage des Theorems ist, dass alle glatten projektiven und absolut irreduziblen Kurven, die über einem lokalen Körper definiert sind, ein semistabiles Modell besitzen, vorausgesetzt, man lässt zu, den Grundkörper durch eine geeignete endliche Erweiterung zu ersetzen. Obwohl bereits der erste Beweis von Deligne und Mumford im theoretischen Sinne als konstruktiv zu bezeichnen ist, erweist es sich in der Praxis als außerordentlich schwierig, diese speziellen Modelle zu finden oder auch nur eine geeignete Körpererweiterung im Sinne des Theorems anzugeben. Das Ziel von [AW11] war, einen alternativen Beweis zu geben, der es ermöglicht, über einen praxistauglichen Algorithmus die semistabile Reduktion von Kurven zu berechnen, zumindest in wichtigen Spezialfällen.

Unser Zugang, ein semistabiles Modell für eine gegebene Kurve zu finden, basiert auf der folgenden Beobachtung: Wenn die betrachtete Kurve als nichtkonstante separable Überlagerung der projektiven Geraden aufgefasst wird und der Grundkörper groß genug ist, garantiert das Theorem über semistabile Reduktion die Existenz eines semistabilen Modells für die projektive Gerade, dessen Normalisierung ein semistabiles Modell für die betrachtete Kurve ist. Ohne auf diese Aussage zurückzugreifen, nehmen wir den umgekehrten Weg und konstruieren sowohl eine hinreichend große Körpererweiterung als auch ein geeignetes Modell der projektiven Geraden, das ein semistabiles Modell für die ursprüngliche Kurve induziert. Die rationalen Komponenten des Modells der projektiven Geraden können dabei so interpretiert werden, dass sie Modifikationen für das Kurvenmodell induzieren; unsere Aufgabe bestand nun darin, geeignete Modifikationen zu finden, die die Singularitäten des Kurvenmodells verbessern. Der entscheidende Punkt ist, dass diese Modifikationen durch rein lokale Betrachtungen der einzelnen Singularitäten gewonnen werden können. Mit Methoden der rigid-analytischen und formalen Geometrie kann das Theorem über die semistabile Reduktion dann zu einer lokalen Aussage reduziert werden, die Überlagerungen der offenen rigid-analytischen Einheitsscheibe betrifft. Mit theoretischen Argumenten ist eine weitere Reduktion auf zwei Spezialfälle möglich, und zwar genügt es, die verbessernden Modifikationen für prim-zyklische Überlagerungen der offenen Scheibe und für prim-zyklische Überlagerungen offener Ringbereiche zu bestimmen.

In beiden Fällen können die verbessernden Modifikationen explizit konstruiert werden – was genau der Zweck der vorliegenden Dissertationsschrift ist. Die Modifikationen werden wie folgt gefunden: Eine Überlagerung der zu betrachtenden Form vom primen Grad p wird durch eine Kummergleichung $y^p = f$ beschrieben, wobei f eine Potenzreihe oder eine Laurentreihe ist. Der entscheidende Schritt besteht nun darin, f sorgfältig durch eine $p$te Potenz ‘so gut wie möglich’ anzunähern; dann kann die gesuchte Modifikation am Newtonpolygon von $f - h^p$ abgelesen werden. Das Hauptwerkzeug, um diese sogenannten hinlänglich genauen Annäherungen zu finden, ist eine Erweiterung der p-Taylorentwicklung, einer Methode, die von Matignon [Mat03] im Zuge seiner Studien des Spezialfalls äquidistanter Geometrie eingeführt wurde. Einer der Gründe, warum sich formale p-Taylorentwicklungen als so nützlich erweisen, besteht darin, dass automatisch eine Körpererweiterung produziert wird, wie sie in der Aussage des Theorems über semistabile Reduktion vorkommt. Folglich können wir

explizit semistabile Modelle für beliebige prim-zyklische Überlagerungen bestimmen. Dies vervollständigt auch den in [AW11] begonnenen Beweis.

Zusätzlich zeigen wir noch, wie unsere Resultate über prim-zyklische Überlagerungen genutzt werden können, um eine beliebige zyklische Überlagerung von Primpotenzgrad zu handhaben. Darüber hinaus demonstrieren wir alle unsere Methoden anhand geeigneter Beispiele. Für einige der umfangreicheren Rechnungen haben wir das Computeralgebra-System Magma [Magma] genutzt; der Quellcode ist an den entsprechenden Stellen im Text abgedruckt.

Schlagwörter: Algorithmische semistabile Reduktion, Prim-zyklische Galois Überlagerungen, Formale p-Taylor Entwicklung

SEMISTABLE REDUCTION OF PRIME-CYCLIC GALOIS COVERS

KAI ARZDORF

Dedicated to authentic moments.

CONTENTS

INTRODUCTION

Main Results. When studying the arithmetic of a curve defined over a local field, one is often interested in having an integral model that has good properties and allows to better understand the behavior of the curve under reduction modulo the finite place of the ground field. By an integral model we mean a normal, flat, and proper scheme over the valuation ring of the local field, having the curve under consideration as generic fiber. In general, we cannot expect to find a smooth model; but at least the badness of occurring singularities can be limited by considering a semistable model—that is, an integral model with absolutely reduced fibers having only ordinary double points as singularities. The well-known theorem of Deligne and Mumford [DM69, Cor. 2.7] ensures that such a semistable model can always be found, provided one allows to replace the base field by a sufficiently large finite extension (see Thm. 1.2 for the precise statement).

Although several different proofs of this theorem exist (some of them making use of rigid analytic methods), it is in practice very hard to determine the semistable model of a given curve, and 'there is no general method to compute the semistable reduction when the residue field is of positive characteristic', as Liu [Liu02, Sect. 10.4.3] states. Note that already the first proof of Deligne and Mumford—make certain Torsion points on the Jacobian rational and compute the minimal regular model—is constructive in a theoretical sense, but this does not seem to yield a practical method except for curves of very small genus. In our recent article with Wewers [AW11], we have presented a new proof, which also exploits rigid analytic methods. But in contrast to most of the proofs published so far, we solely rely on local arguments that do not involve the global geometry of the curve (for instance, its Jacobian). This makes the proof in

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large parts constructive and allows to come up with a practical algorithm to compute the semistable reduction of given curves, at least in important special cases (including cases that were inaccessible before).

Our approach to finding a semistable model for a given curve is based on the following observation: when the curve is considered as a non-constant separable cover of the projective line and if the base field is large enough, the Semistable Reduction Theorem guarantees the existence of a semistable model for the projective line whose normalization is a semistable model for the curve under consideration. Without relying on this fact, we take the reversed approach and construct both a large enough extension of the base field and a suitable model for the projective line that induces a semistable model for the initial curve. The rational components of these projective line models can be interpreted as inducing modifications of the original curve model; we therefore have to find the right modifications leading to a semistable model.

More precisely, the algorithm described in loc.cit. proceeds as follows. Starting with the standard smooth model for the projective line, normalization yields a model for the initial curve that will probably be highly singular. When working over a sufficiently large field, the corresponding special fiber will at least be reduced (this, however, is based on a theoretical result by Epp [Epp73], and in practice, finding a suitable field extension is one of the major problems; nevertheless, in the important case of prime-cyclic coverings to be introduced below, we will be able to explicitly construct a well-suited field). Even when the special fiber is reduced, there still might be bad singularities on it; our task is thus to find modifications that improve these singularities. The crucial point is that these modifications can be derived from purely local considerations by examining each occurring singularity individually (that is, by completion in the respective points). Namely, using methods from rigid analytic and formal geometry, one is lead to study covering maps between open analytic curves (rigid analytic spaces providing a non-archimedean analog of open Riemann surfaces with finitely many holes). More precisely, coverings of the rigid analytic open unit disk have to be studied. The modifications improving the singularities of the curve model then correspond to certain closed subdisks that are smallest in the sense of capturing 'just enough' information of the covering, with 'enough' meaning, there must be no loss of genus when restricting the covering to these smaller disks (which are then called exhausting). By considering an appropriate numerical invariant measuring the badness of occurring singularities, it can be seen that it suffices to repeat the above steps a finite number of times to end up with a semistable model for the initial curve.

Finding those minimal disks is the most delicate part of the whole process and highly non-trivial; knowing how to find these disks means proving the Semistable Reduction Theorem! In general, it is not clear at all how the disks have to be chosen to be minimal exhausting. When the covering has a solvable Galois group[1], induction on the group order can be used to reduce to the case of prime-cyclic Galois covers of the open unit disk. To make the induction step work, however, we also need to handle prime-cyclic coverings of open annuli; in this situation, the objective is to find the so-called maximal separating boundary domain, which generalizes the notion of the minimal exhausting disk. Both these cases have still to be settled—and this is precisely the purpose of the present dissertation. The case of a non-solvable Galois group can be reduced to

[1] As far as the Semistable Reduction Theorem is concerned, we can assume all occurring coverings to be Galois.

the solvable case by considering certain stabilizer subgroups (though at present, this reduction step is not practical).

In this dissertation, we address the above mentioned two main cases and study prime-cyclic coverings of the open unit disk and of open annuli. In each case, we explicitly construct the maximal separating boundary domain (Thms. 2.1 and 4.2), thereby also providing the final ingredient for our new proof of the Semistable Reduction Theorem. The construction is done as follows: A covering as above and of prime degree p is given by a Kummer type equation $y^p = f$, with f a power series or a Laurent series. The crucial step is to carefully approximate f by a pth power h^p 'as good as possible'; then the equation defining the covering can be rewritten in such a way that not only an algebraic description of the model and its special fiber is attained, but the equation is also stable in the sense that it can be used for describing the induced coverings on subdisks or subannuli. The consequence is that the thickness of the maximal separating boundary domain can be read off from the Newton polygon of $f - h^p$, as Props. 2.33 and 4.12 state.

The main tool for obtaining these so-called sufficiently precise approximations (Defs. 2.25 and 4.5) is by use of formal p-Taylor expansions (Defs. 2.9 and 4.13), a generalization of a method introduced by Matignon [Mat03] in his studies of the equidistant subcase. It takes some work before we can establish with Props. 2.31 and 4.6 the existence of sufficiently precise approximations. In the disk case, this is due to the fact that the center point of the minimal exhausting disk is a priori unknown. Though finding a suitable center for the maximal separating boundary domain is no problem in the case of open annuli, this situation is still more complicated in that a variety of different cases can occur due to the richer algebraic structure of analytic functions on open annuli; some special cases even require to utilize an additional approximation algorithm. One of the reasons why formal p-Taylor expansions prove so useful is that a field extension as in the statement of the Semistable Reduction Theorem is automatically produced (thereby also making the theoretical result of Epp effective in the prime-cyclic case). The result is that we can explicitly calculate semistable models for arbitrary cyclic coverings of prime degree; see Rems. 2.36 and 4.20. Note that this also completes the new proof of the Semistable Reduction Theorem started in [AW11].

With the results from above, it also becomes possible to calculate semistable models for an arbitrary cyclic covering of prime power degree. This is by splitting the covering into successive prime-cyclic substeps, which can then be handled with our methods for prime-cyclic coverings of the open unit disk and of open annuli.

The dissertation contains several elaborate examples illustrating all our methods. First, an equidistant prime-cyclic covering of the projective line is studied in detail; see Sect. 3. The calculations are non-trivial and the involved field extensions are complicated; yet we can work out the minimal semistable model and the monodromy action on the stable reduction. We have used the computer algebra system MAGMA [Magma] to support us in the calculations; the source code can be found at the respective places in the text, making it possible to reconstruct all details of our computations. As a complement to this example, we study in Sect. 4.6 a prime-cyclic covering not being of equidistant geometry. This particular curve occurred in the literature before, and using our new methods, we are able to determine both its monodromy extension and its stable reduction. In a third example, our methods are applied to the case of a cyclic étale disk covering of prime-square degree (a situation that could not be handled before), and we construct the corresponding stable model; see Sect. 5.2.

Structure. In the first section, we review our algorithmic approach to semistable reduction, giving necessary background information and recalling results from rigid analytic and formal geometry; we state the two main cases that have yet to be settled. The second section deals with the case of prime-cyclic étale Galois covers of the open unit disk; the detail study of an appropriate example follows in the third section. The ramified situation is treated in the fourth section, where more generally prime-cyclic étale covers of open annuli are studied. In the fifth and final section, we show how to reduce the general case of a cyclic Galois cover of prime power degree to the prime-cyclic case.

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1. Algorithmic Approach to Semistable Reduction

In this section, we explain our approach to the Semistable Reduction Theorem and the underlying algorithm. Taking a relative viewpoint and using the language of rigid analytic geometry, the Semistable Reduction Theorem can be reduced to a more local statement that is better suited for algorithmic purposes. We identify two main cases that need to be settled for our proof to become complete. Along the way, we recall a number of facts from rigid analytic and formal geometry, giving auxiliary results as needed.

Full details concerning the proof and the algorithm can be found in our recent paper with Wewers [AW11]; the present dissertation is concerned with an in-depth study of the two mentioned key cases, see Sects. 2 and 4.

1.1. **Basic Notation.** We mostly follow the notation used by Wewers and the author in [AW11]. So K shall denote a complete non-archimedean field, with the valuation ring and its maximal ideal denoted by R and $\mathfrak{m}$, the residue field by $k := R/\mathfrak{m}$, and the valuation by v. A uniformizer of R shall be denoted by π. With the Semistable Reduction Theorem in mind, we can assume k to be algebraically closed, cf. [Liu02, Lem. 10.4.5]. In later sections, we will assume K to have mixed characteristic $(0, p)$ (one should have fields like $\mathbb{Q}_p^{\mathrm{nr}}$, the maximal unramified extension of the p-adic numbers, in mind). In this case, we always assume the valuation to be normalized such that $v(p) = 1$. For now, however, there is no further restriction on the characteristic.

We also fix an algebraic closure K^{ac} of K and extend the valuation v to K^{ac} (in a unique way); the corresponding valuation ring is denoted by $(R^{\mathrm{ac}}, \mathfrak{m}^{\mathrm{ac}})$. In the following, we will often deal with (finite) field extensions of K; these are always considered as subfields of K^{ac} and provided with the uniquely extended valuation.

The value group of K is denoted by $v(K^\times)$; the one of the fixed algebraic closure by $v(K^{\mathrm{ac}\times})$. Though in the context of rigid analytic geometry it is more common and convenient to work with an absolute value $|\cdot| := q^{-v(\cdot)}$ (where $q > 1$ denotes an arbitrary but fixed real number) instead of the exponential valuation v, we only work with the latter. This is to avoid any conflicts that could arise when working with inequalities and mixing absolute values and valuations.

1.2. **Models for Curves.** We recall the definitions for various models of curves and give a first version of the Semistable Reduction Theorem (Thm. 1.2); we will later derive more relative and local formulations (Thms. 1.15 and 1.24).

1.2.1. *Semistable Models.* Let Y be a smooth projective and absolutely irreducible curve over K. A *model* for Y is a normal flat and proper R-scheme Y_R having Y as generic fiber: $Y_R \otimes_R K \cong Y$. The special fiber of Y_R—denoted by $\bar{Y}$—is proper and (due to the flatness assumption) of pure dimension one; by Zariski's main theorem, it is connected.

Definition 1.1. A model Y_R for a smooth projective and absolutely irreducible K-curve Y is called *semistable* if its fibers are absolutely reduced and have at most ordinary double points as singularities.

If a semistable model for Y exists, Y is said to have *semistable reduction*. Often a semistable model only exists after finitely enlarging the base field; in this case, the field extension is said to *realize* the semistable reduction, and we say that Y has *potentially semistable reduction*. The well-known theorem of Deligne and Mumford [DM69, Cor. 2.7] states that all K-curves have potentially semistable reduction.

Theorem 1.2 (Semistable Reduction Theorem)**.** *Let Y be a smooth projective and absolutely irreducible curve over the local field K. There exists a finite field extension L/K such that $Y \otimes_K L$ has semistable reduction. Moreover, L/K can be chosen to be separable.*[2]

For the rest of this section, we assume that Y is a curve of genus $g_Y \geq 2$; in this case, the above results can be strengthened. Let the field extension L/K realize the semistable reduction of Y. There exists a *stable model* for $Y \otimes_K L$, that is, a semistable model whose fibers are of arithmetic genus ≥ 2 and which satisfy the additional combinatorial property that each of their rational components intersects the other components in at least three points. Moreover, this model is unique; it is the unique minimal semistable model in the sense that it results from any given semistable model by contraction of all superfluous rational components. The special fiber of the stable model does not depend on the field extension over which the stable model is realized; it is called the *reduction* of Y. See [Liu02, Def. 10.3.27 and Thm. 10.4.3].

Let L/K be a Galois extension realizing the stable reduction of Y; denote by Y^{ss} the stable model of $Y \otimes_K L$ and by $\bar{Y}$ its special fiber, the reduction of Y. The Galois group $\mathrm{Gal}(L/K)$ acts tautologically on $Y \otimes_K L$. Due to the uniqueness of the stable model, the action extends to the so-called *monodromy action*

$$\rho : \mathrm{Gal}(L/K) \to \mathrm{Aut}_k(\bar{Y})$$

on the reduction of Y; see [Liu02, Cor. 10.3.37].

Corollary 1.3. *Still keeping the assumption $g_Y \geq 2$, there is a unique minimal field extension $K^{\min}/K$ over which a stable model of Y can be defined, in the sense that any extension realizing the stable reduction of Y contains $K^{\min}$. This extension is Galois and called the* monodromy extension*; it is characterized by the fact that the monodromy action of* $\mathrm{Gal}(K^{\min}/K)$ *on $\bar{Y}$ is faithful.*

Proof. This is essentially [Liu02, Thm. 10.4.44]. □

[2] If K has characteristic zero, this is automatic of course.

Note that the monodromy extension can be obtained as the fixed field $L^{\ker\rho}$, where L/K denotes any Galois extension realizing the stable model of Y and where ρ is the corresponding monodromy action.

The above results concerning the monodromy require the residue field of K to be algebraically closed, which is what we always assume: see our notation and conventions from Sect. 1.1. Sometimes, however, one is interested in curves defined over base fields that do not have an algebraically closed residue field (as in the examples treated in Sects. 3 and 4.6, where we deal with curves defined over $\mathbb{Q}_3$ resp. $\mathbb{Q}_7$). If this is the case, one might have to pass to a sufficiently large unramified extension for the monodromy action to be defined and for the monodromy extension to be unique. Yet this is no serious problem, and for our notions to be consistent with this more general setting, we make the following definition.

Definition 1.4. Let Y be a smooth projective and absolutely irreducible curve over the local field K_0, and let $g_Y \geq 2$. Assume that the field extension L_0/K_0 realizes the stable reduction of Y. Then L_0/K_0 is called *minimal*, if base change to the maximal unramified extension gives the monodromy extension; that is, if $L_0K_0^{\mathrm{nr}}/K_0^{\mathrm{nr}}$ is the monodromy extension of $Y \otimes_{K_0} K_0^{\mathrm{nr}}$.

As mentioned in the introduction, it is generally rather difficult to explicitly determine a semistable model for a given curve, or the field extension needed to realize the stable reduction. This is despite the fact that—in theory—we know how to obtain a semistable model: the original proof of Deligne and Mumford shows that the stable reduction of a K-curve Y is realized over a field extension L/K, which makes the 3- and 4-torsion points on the Jacobian of Y rational. A semistable model can then be obtained by resolving singularities: the minimal regular model for $Y \otimes_K L$ will be semistable. This approach, however, seems not to be practical. Neither is it easy to determine the Jacobian of a given curve (at least when its genus is not particularly small), nor is it easy to then deduce an appropriate field extension for making those torsion points rational. Further more, the extension obtained in this way would be very large—probably much larger than needed and most likely too large for doing actual computations. Yet the above approach of determining a semistable model by calculating a regular model requires the knowledge a sufficiently large field in advance! This is because regularity is not stable under base change and all work of computing a regular model (which in itself is not a trivial task) would be in vain if a further field extension was needed—for instance, because the special fiber of the regular model was not reduced.

It should be reasonably clear by now that different methods are required for actually computing semistable reductions, and this is what our new proof accomplishes: our approach is based on local considerations and can be turned into a practical procedure, which can be followed step by step. This will become clear in the sections to follow, also by means of the explicitly computed examples.

1.2.2. *Permanence.* Though models are normal by definition, their special fibers usually contain singular points. Enlarging the base field, we can at least assume the special fibers to be reduced. This is essentially due to a result by Epp [Epp73]; the details of the reasoning can be found in [AW11, Prop. 2.3].

Proposition 1.5. *Let Y_R be an R-model for the smooth projective and absolutely irreducible K-curve Y. There exists a finite field extension L/K such that the special fiber* *of the normalized base change $Y_S := (Y_R \otimes_R S)^\sim$—which is an S-model for $Y \otimes_K L$—is reduced. Furthermore, if L'/L is any field extension, then the usual base change $Y_S \otimes_S S'$ is normal with reduced special fiber. (Here, S and S' denote the integral closures of R in L and L', respectively.)* □

Hence, in the context of the Semistable Reduction Theorem, we may always assume that the special fibers of all occurring models are reduced. We call models with this property *permanent* (this notion goes back to Raynaud).

Convention 1.6. We always allow the base field K to get replaced by a finite field extension (considered as a subfield of K^{ac}). In particular, by Prop. 1.5, we can assume all models we work with to be permanent, that is, to have reduced special fiber. Of course, when doing actual calculations, one has to keep track of the field extensions involved.

Though in theory—and in particular, in our proof of the Semistable Reduction Theorem—we can always assume to work with permanent models, in practice, it is usually not clear which field extension is required to make a given model permanent. This is because Epp's result has not been made explicit yet; with respect to doing actual computations, this has to be kept in mind. We will later see that in the important case of prime-cyclic coverings (treated in Sects. 2 and 4), we can make all steps of our algorithm explicit and do not have to rely on theoretical assumptions; cf. Rems. 2.36 and 4.20.

1.2.3. *Modifications and Singularity Measure.* Let Y_R be a permanent model for the smooth projective and absolutely irreducible K-curve Y. If Y_R is not semistable, there are finitely many non-nodal singularities of the special fiber. Our aim is to modify Y_R in such a way that these singularities improve. We will make this more precise though we will not go into the details, as this paper is mainly concerned with the local situation to be introduced in later sections; for full details, the reader should consult [AW11, Sects. 2.3 and 2.4].

For a closed point $y \in \bar{Y}$ on the special fiber of Y_R, we define the numbers

$$\delta_y := \dim_k(p_*\mathcal{O}_{(\bar{Y})^\sim}/\mathcal{O}_{\bar{Y}}) \quad \text{and} \quad m_y := |p^{-1}(y)|\,,$$

where $p : (\bar{Y})^\sim \to \bar{Y}$ denotes the normalization morphism of the special fiber. These numbers provide a measure for the badness of a singularity: $y \in \bar{Y}$ is smooth if and only if $\delta_y = 0$; it is an ordinary double point precisely when $\delta_y = 1$ and $m_y = 2$. We always have $\delta_y \geq m_Y - 1$, and the arithmetic genus of $\bar{Y}$ can be related to the genus of its components and the sum of the delta-numbers of its closed points; see [Liu02, Props. 7.5.4 and 7.5.15].

A *modification* of Y_R is a birational R-morphism $f : Y_R' \to Y_R$, where Y_R' denotes another R-model of Y; note that f is the identity on the generic fiber. The modification is called *permanent* if Y_R' is permanent; finitely enlarging the base field K, any given modification becomes permanent, cf. Conv. 1.6. The subset of $\bar{Y}$ where f is not an isomorphism is called the *center* of f. An irreducible component $W \subset \bar{Y}'$ is *exceptional* if $f(W)$ is a closed point. The union of all exceptional components forms the *exceptional divisor*; the union of all irreducible components that are not exceptional is called the *strict transform* of $\bar{Y}$.

Our proof of the Semistable Reduction Theorem requires to consider a special kind of permanent modifications, which we call *simple*: every exceptional component of a simple modification has to intersect the strict transform, and every such point of intersection has to be an ordinary double point of $\bar{Y}'$. A simple modification is considered an *improvement* at the singular point $y \in \bar{Y}$ if for every closed point $y' \in f^{-1}(y)$ not on the strict transform of $\bar{Y}$

$$\text{either} \quad \delta_{y'} < \delta_y \quad \text{or} \quad \delta_{y'} = \delta_y,\ m_{y'} > m_y$$

holds true.

1.2.4. *Algorithm.* We can now explain the idea behind our algorithmic approach to finding the semistable reduction of a given curve Y.

We start with some permanent model Y_R for Y. If Y_R is not semistable, its special fiber $\bar{Y}$ contains a finite number of non-nodal singularities. The genus of Y gives an upper bound for the sum of the delta-numbers on $\bar{Y}$. Applying an improving modification—assuming for the moment that we can always find one—the respective delta-numbers get smaller (or at least corresponding m-numbers get larger, but this can happen only a finite number of times). Hence, after finitely many of these modifications, all delta-numbers are zero or one (in the latter case, with the corresponding m-numbers being two)—that is, all bad singularities have been resolved and a semistable model is obtained.

To make our algorithm work, we need to see that improving modifications for a given model with bad singularities always exist; what is more, we are interested in a practical way to determine the respective modifications. Our local approach deals with this issue by studying the completed local rings of singular points using methods from rigid analytic and formal geometry, which are explained in the next section.

1.3. **Notions from Rigid Analytic Geometry.** In this section, we recall some ideas and concepts from rigid analytic and formal geometry; we will not have to use deep theorems of these fields, but rather use the language specific to rigid analytic geometry as a device that allows to state our results in a clear, succinct way.

1.3.1. *Specialization Map and Tube.* As before, let Y denote a smooth projective and absolutely irreducible K-curve and let Y_R be a permanent R-model for Y. As explained in [FvdP04, Example 4.3.3], one can associate a rigid analytic K-space Y^{rig} to Y (in a functorial way); the underlying set of points of Y^{rig} is just the set of closed points of Y.

Given a point of Y^{rig}—that is, a closed point $y \in Y$—the scheme theoretic closure $\overline{\{y\}}$ in Y_R intersects the special fiber in a unique closed point $\bar{y} \in \bar{Y}$, the *specialization* of y. Indeed, $\overline{\{y\}}$ is finite and irreducible over R, and thus a local scheme since the complete field K is in particular Henselian. The resulting map

$$\mathrm{sp}_{Y_R} : Y^{\mathrm{rig}} \to \bar{Y}$$

is called the *specialization map* of Y with respect to the model Y_R; of course, it heavily depends on the choice of the model.

The specialization map is surjective; see [Liu02, Cor. 10.1.38]. Given a locally closed subscheme $Z \subset \bar{Y}$, the preimage

$$]Z[_{Y_R} := \mathrm{sp}_{Y_R}^{-1}(Z) \subset Y^{\mathrm{rig}}$$

is not a scheme but can be endowed with the structure of a rigid analytic space; this is because the preimage is open in the Grothendieck topology for Y^{rig}. It is hence a smooth rigid analytic K-space, called the *formal fiber* or *tube* of Z in Y_R; see [Bos77] or [Ber96, Sect. 1] for more details. Furthermore, if Z is connected, then $]Z[_{Y_R}$ is connected as well, as follows from [Ber96, Prop. 1.3.3].

1.3.2. *Formal Models.* Let $\mathscr{Y} := Y_R|_Z^\wedge$ denote the formal completion of the model Y_R along the subscheme Z. As a formal scheme, $\mathscr{Y}$ does no longer have a generic fiber in the classical sense. However, one can construct and associate a rigid analytic space $\mathscr{Y}_K$ to $\mathscr{Y}$ serving as a generic fiber; this space is canonically isomorphic to the tube $]Z[_{Y_R}$ as defined above, and one has an isomorphism

$$\Gamma(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}) \xrightarrow{\sim} \Gamma(]Z[_{Y_R}, \mathring{\mathscr{O}}_{Y^{\mathrm{rig}}}) \tag{1.1}$$

between the global sections of the structure sheaf on $\mathscr{Y}$ and the sections over $]Z[_{Y_R}$ of analytic functions bounded by 0. Recall that an analytic function f on the rigid analytic space Y^{rig} is called *bounded by* 0 or *zero-bounded* if for all $y \in Y^{\mathrm{rig}}$, we have $v_y(f(y)) \geq 0$ (with v_y denoting the valuation on the respective residue field).

We comment on the construction in the case of an open affine Z, that is, when Z is the spectrum of $\bar{B} = B/(\pi)$ and $\mathscr{Y} = \operatorname{Spf} B$. In this situation, the ring B is flat, topologically of finite type over R, and complete with respect to the π-adic topology. Then the base change $B_K := B \otimes_R K$ is an affinoid K-algebra, and the corresponding affinoid domain $\mathscr{Y}_K := \operatorname{Spm} B_K$ gives the generic fiber associated to $\mathscr{Y}$; this is the classical construction due to Raynaud [Ray74]. As remarked above, the so-constructed generic fiber coincides with the tube of Z along Y_R. Using isomorphism (1.1), B can be recovered as the ring

$$\mathring{B}_K := \{ f \in B_K \mid v_y(f(y)) \geq 0 \text{ for all } y \in \mathscr{Y}_K \}$$

of analytic functions on $\mathscr{Y}_K$ bounded by 0; that is, $B = \mathring{B}_K$. It follows that $\bar{B}$ describes the canonical reduction of the affinoid domain $]Z[_{Y_R}$ in the sense of [FvdP04].

When Z is not an open affine, the involved rings are no longer topologically of finite type over R, and more subtle arguments have to be given to get the associated rigid space in this more general situation; see [Ber96, Sects. 0 and 1] and [dJ95, Sect. 7].

1.3.3. *Open Analytic Curves.* When $Z = \{y\} \subset \bar{Y}$ consists of a single closed point of the special fiber, the tube $\mathsf{Y} :=]y[_{Y_R}$ is called the *residue class* of $y \in \bar{Y}$ (with respect to the model Y_R). By the results from Sect. 1.3.2 above, Y can be identified with the generic fiber associated to the formal R-scheme $\operatorname{Spf} B$, where $B := \mathscr{O}_{Y_R,y}^\wedge$ denotes the completion of the local ring at $y \in Y_R$; moreover, by isomorphism (1.1), B can be identified with the ring of zero-bounded analytic functions on Y. As a consequence, the rigid analytic space Y—that is, the residue class of the point $y \in \bar{Y}$—only depends on the completion of the model Y_R at its point y. In this sense, the study of residue classes is of a local nature.

Definition 1.7. An *open analytic curve* over K is a rigid analytic K-space Y that can be realized as a residue class $]y[_{Y_R}$, where Y_R is a permanent model for a smooth projective and absolutely irreducible K-curve Y, and $y \in \bar{Y}$ is a closed point of the special fiber. The formal R-scheme $\mathscr{Y} = \operatorname{Spf} B$, where $B = \mathring{\mathscr{O}}_{\mathsf{Y}}$ denotes the ring of zero-bounded analytic functions on Y, is called the *canonical formal model* of Y; we denote by $K(\mathsf{Y}) := \operatorname{Frac} B$ the corresponding fraction field.

Remark 1.8. By the facts stated in Sect. 1.3.1, open analytic curves are smooth and absolutely connected, hence absolutely irreducible. As a consequence, the corresponding ring of zero-bounded analytic functions is an integral domain (which is also complete and normal).

Remark 1.9. Recall that by Conv. 1.6, we always assume all models we work with to be permanent; in particular, we can use them for the definition of open analytic curves.

Remark 1.10. One should think of open analytic curves as a non-archimedean analog of open Riemann surfaces with finitely many holes. The drawings throughout this paper, which illustrate new notions or certain set-ups, are kept in accordance with that viewpoint; for instance, see Figs. 1.2 and 1.3.

Remark 1.11. The concept of open analytic curves can be studied in greater generality, as, for example, Wewers [Wew05] does. Nevertheless, for our practical purposes, the concrete definition from above is sufficient.

1.3.4. *Open Unit Disk and Open Annuli.* The most basic examples for open analytic curves are the following.

Definition 1.12. An *open disk* is a rigid analytic K-space isomorphic to the standard rigid analytic open unit disk

$$\{x \in \mathbb{A}^1_K \mid 0 < v(x)\} ;$$

an *open annulus of thickness* ϵ is isomorphic to

$$\{x \in \mathbb{A}^1_K \mid 0 < v(x) < \epsilon\} ,$$

for some $\epsilon > 0$ in the value group $v(K^\times)$. Confer Fig. 1.1.

The next lemma shows that both open disks and open annuli are indeed open analytic curves (that is, occur as residue classes of permanent models for smooth K-curves). The lemma also shows that open analytic curves in some sense indicate how smooth their canonical reduction is.

Lemma 1.13. *Let Y_R be a permanent model for the smooth projective and absolutely irreducible K-curve Y, and denote by $\mathsf{Y} :=]y[_{Y_R}$ the residue class associated to a closed point $y \in \bar{Y}$ on the special fiber of Y_R. Then the residue class Y is*

1. *an open disk if and only if $y \in \bar{Y}$ is a smooth point,*
2. *an open annulus if and only if $y \in \bar{Y}$ is an ordinary double point.*

In particular, the open unit disk resp. the open annulus of thickness ϵ can be realized as the residue class corresponding to the formal scheme $\operatorname{Spf} R[\![t]\!]$ resp. $\operatorname{Spf} R[\![s,t \mid st = a]\!]$ (with $a \in K^\times$ of valuation $v(a) = \epsilon$). See the illustration in Fig. 1.1.

Proof. This is [AW11, Prop. 3.4]. □

1.3.5. *Boundary Points and Rank-Two-Valuations.* We have remarked in Rem. 1.10 that open analytic curves should be seen as a non-archimedean analog of open Riemann surfaces. The notion of boundaries arising under this viewpoint is of interest because the boundaries can be related to certain valuations encoding useful information on the respective rigid analytic space.

Definition 1.14. Let Y be an open analytic curve with corresponding formal model $\operatorname{Spf} B$. A *boundary point* of Y is a generic point of the special fiber $\operatorname{Spec} B/(\pi)$. The (finite) set of all boundary points of Y is denoted $\partial\mathsf{Y}$.

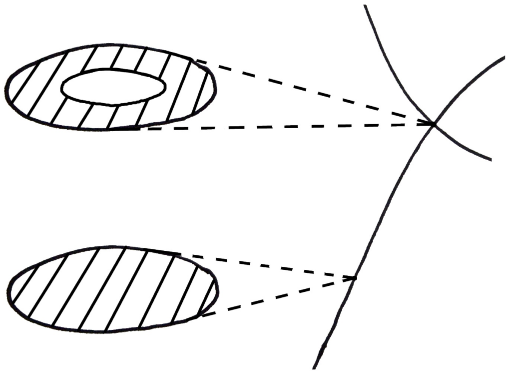

FIGURE 1.1. The residue class of a closed point of the special fiber is an open disk resp. an open annulus if and only if the point is smooth resp. an ordinary double point.

Assume that Y is realized as the residue class $]y[_{Y_R}$ of the closed point $y \in \bar{Y}$ on the special fiber of a curve Y with permanent model Y_R. Then B is the completion of the local ring $\mathcal{O}_{Y_R,y}$. It follows that a boundary point of Y corresponds to a local branch of the special fiber passing through y. In particular, an open disk has precisely one boundary point, while an open annulus has two boundary points (which is in accordance with our intuitive conception of boundaries under the viewpoint of open Riemann surfaces); cf. Fig. 1.1.

A boundary point $\xi \in \partial\mathsf{Y}$ corresponds to a height-one prime ideal $\mathfrak{q}$ of B containing the uniformizer π of the ground field K; the boundary point thus gives rise to a discrete valuation $v_{\mathfrak{q}}$ on $\operatorname{Frac} B$. In geometric terms, it corresponds to the sup-norm (here, under the valuation-theoretic point of view, to the inf-valuation) at the respective boundary of the open analytic curve Y. As we have $\pi \in \mathfrak{q}$, the restriction of $v_{\mathfrak{q}}$ to K is equivalent to the valuation v on K; we assume $v_{\mathfrak{q}}$ to be normalized such that equality holds true (in the case of mixed characteristic $(0, p)$, this is done by demanding $v_{\mathfrak{q}}(p) = 1$, cf. Sect. 1.1). Due to the fact that Y is defined in terms of a permanent model, its special fiber is reduced. As explained in [AW11, Proof of Prop. 2.3], this implies that the extension of discrete valuations $v_{\mathfrak{q}}/v$ is weakly unramified; that is, the two value groups of $v_{\mathfrak{q}}$ and v coincide. Furthermore, note that the residue field $k(\mathfrak{q})$ of $v_{\mathfrak{q}}$ is isomorphic to the discretely valued field $k(\!(\bar{t})\!)$, as $k(\mathfrak{q})$ is the function field of the local branch corresponding to ξ.

Denoting the discrete valuation on $k(\mathfrak{q})$ by $\#_{\mathfrak{q}}$ (normalized such that $\#_{\mathfrak{q}}(\bar{t}) = 1$) and composing the two valuations $v_{\mathfrak{q}}$ and $\#_{\mathfrak{q}}$ in the sense of [ZS60, Sect. 10], we get a *rank-two-valuation* v_ξ on $\operatorname{Frac} B$:

$$v_\xi : \operatorname{Frac} B \to \mathbb{Q} \times \mathbb{Z}, \quad f \mapsto (v_{\mathfrak{q}}(f), \#_{\mathfrak{q}}(\overline{f/a})),$$

where $a \in K$ is any element having valuation $v(a) = v_{\mathfrak{q}}(a) = v_{\mathfrak{q}}(f)$. We denote the second component of the rank-two-valuation by $\#_\xi$; this is a map from $\operatorname{Frac} B$ to $\mathbb{Z}$ (but not a valuation). In the case where Y is an open disk and $B = R[\![t]\!]$, this is precisely the Weierstraß order; that is, $\#_\xi(f)$ is the degree of the distinguished polynomial $P \in R[T]$ in the Weierstraß decomposition $f = bPu$ (with $b \in R$ and $u \in R[\![t]\!]^\times$).

Later on, we will use these valuations to formulate some criteria that enable us to recognize when open analytic curves are open disks or open annuli; see Lems. 1.28 and 1.30.

1.4. **Covering Maps.** Though the Semistable Reduction Theorem is concerned with describing the reduction of a single curve, it will prove useful to take up a relative viewpoint and interpret the curve under consideration as a finite cover of the projective line. The underlying idea is that in this situation, the Semistable Reduction Theorem guarantees the existence of a semistable model for the projective line, the normalization of which is a semistable model for the initial curve—provided the base field is large enough. Without relying on this fact, we are going to take the reversed approach and construct both a sufficiently large field extension and a semistable model for the projective line, which induces a semistable model for the initial curve; cf. Thm. 1.15. As we will see below, we can use the rigid analytic methods introduced in Sect. 1.3 to address these problems in a more local setting; cf. Thm. 1.24.

1.4.1. *Relative Viewpoint.* Let Y and X be smooth projective and absolutely irreducible K-curves. Let $\Phi: Y \to X$ be a finite covering map, that is, a finite and flat K-morphism.[3] Then, for any model X_R of X, the normalization Y_R of X_R in the function field $K(Y)$ is a model of Y, and we get a corresponding morphism $\Phi_R: Y_R \to X_R$ between those models, which in turn restricts to a morphism on the special fibers, $\bar{\Phi}: \bar{Y} \to \bar{X}$. Recall that by Conv. 1.6, we can assume both X_R and Y_R to be permanent.

Suppose that we want to determine the semistable reduction of Y. Because the function field $K(Y)$ is of transcendence degree one over K, we can choose an element $t \in K(Y)$ such that $K(Y)$ is finite separable over the rational function field $K(t)$. We interpret the latter as the function field of the projective line $X := \mathbb{P}^1_K$. In geometric terms, the choice of t corresponds to a finite separable covering map $\Phi: Y \to X$. Then the Semistable Reduction Theorem in the formulation of Thm. 1.2 is an immediate consequence of the following relative version Thm. 1.15 (both versions are actually equivalent). It is the relative version that our algorithm is based on.

Theorem 1.15. *Let Y be a smooth projective and absolutely irreducible K-curve together with a finite covering map $Y \to X$ to the projective line $X = \mathbb{P}^1_K$. After a finite extension of the base field K (if necessary), there exists a semistable model X_R of X such that the normalization Y_R of X_R in the function field $K(Y)$ gives a semistable model for Y.*

The proof is based on the following idea, which is explained in detail in [AW11, Sects. 2.6 and 2.7]: We start with some semistable model X_R of the projective line (for instance, with the standard smooth model $\mathbb{P}^1_R$) and its corresponding normalization in $K(Y)$, denoted by Y_R. We can assume both models to be permanent, and also that Y_R is not semistable (otherwise, we were finished). A modification of X_R—that is, a birational morphism $X'_R \to X_R$—induces by normalization in $K(Y)$ a morphism $Y'_R \to Y_R$, which is seen to be a modification of Y_R. The goal is thus to find suitable modifications of X_R that induce improving modifications of Y_R; a finite number of these improvement steps would then suffice to end up with a semistable model for Y, as was explained in Sect. 1.2.4. The insight, that the modifications we are looking for can be deduced from purely local considerations, is the heart of our proof and will be explained in the sections to follow.

[3]Since the curves we deal with are smooth and absolutely irreducible, we could equally well demand Φ to be finite and dominant.

In any case, it suffices to prove Thm. 1.15 under an additional assumption, as Rem. 1.16 shows. We introduce the necessary terms: a finite cover $Y \to X$ between smooth projective and absolutely irreducible K-curves is called a *G-Galois cover* if there is an action of the finite group G on Y such that X is the quotient of Y by G; this is equivalent to the field extension $K(Y)/K(X)$ being Galois with Galois group G.

Remark 1.16. In the statement of Thm. 1.15, we can assume the covering $Y \to X$ to be Galois (see [AW11, Prop. 2.11] for the reasoning).

1.4.2. *Covers of Open Analytic Curves.* Before we can pass from the global to the local setting, we need to introduce some notions for the local situation.

Let $\phi : \mathsf{Y} \to \mathsf{X}$ be a finite and flat morphism between rigid analytic curves. In this situation, ϕ is determined by the ring extension B/A, where $B := \mathring{\mathcal{O}}_{\mathsf{Y}}$ and $A := \mathring{\mathcal{O}}_{\mathsf{X}}$ are the respective rings of zero-bounded analytic functions; see [dJ95]. Since A and B are normal integral domains (see Rem. 1.8), B can be recovered as the integral closure of A in the field extension $K(\mathsf{Y})/K(\mathsf{X})$ (with the usual notation $K(\mathsf{Y}) = \operatorname{Frac} B$ and $K(\mathsf{X}) = \operatorname{Frac} A$ for the rigid analytic function fields).

Definition 1.17. Let the situation be as above. Then ϕ is called a *finite covering map* if B/A is finite. We speak of a *G-Galois cover* if, moreover, $K(\mathsf{Y})/K(\mathsf{X})$ is Galois with Galois group G; in this case, G acts on Y such that X is the quotient of Y by G, and $A = B^G$ can be recovered as the G-invariants of B.

Remark 1.18. A finite covering $\mathsf{Y} \to \mathsf{X}$ of open analytic curves induces a surjective map $\partial\mathsf{Y} \to \partial\mathsf{X}$ between their boundary points: for $\xi \in \partial\mathsf{Y}$, the restriction of v_ξ to $K(\mathsf{X})$ coincides with the rank-two-valuation coming from a uniquely determined boundary point $\eta \in \partial\mathsf{X}$. In other words, every boundary point of Y lies above precisely one boundary point of X.

1.4.3. *From Global to Local.* Let $\Phi : Y \to X$ be a finite G-Galois cover of smooth projective and absolutely irreducible K-curves. Denote by X_R a model for X, and let Y_R be the corresponding normalization in $K(Y)$; this gives a finite morphism $Y_R \to X_R$. As always, we can by Conv. 1.6 assume those models to be permanent. The covering map Φ induces a finite morphism of corresponding rigid analytic spaces, $\Phi^{\mathrm{rig}} : Y^{\mathrm{rig}} \to X^{\mathrm{rig}}$.

Let $y \in \bar{Y}$ be a closed point on the special fiber of Y_R and let $x := \Phi_R(y) \in \bar{X}$ be its image point, which is a closed point on the special fiber of X_R. We denote by $\mathsf{Y} :=]y[_{Y_R} \subset Y^{\mathrm{rig}}$ and $\mathsf{X} :=]x[_{X_R} \subset X^{\mathrm{rig}}$ the corresponding formal fibers; by Def. 1.7, these are determined by the complete local rings $\mathcal{O}^\wedge_{Y_R,y}$ and $\mathcal{O}^\wedge_{X_R,x}$, respectively. The rigid analytic map Φ^{rig} restricts to a map

$$\phi = \Phi^{\mathrm{rig}}|_{\mathsf{Y}} : \mathsf{Y} \to \mathsf{X}$$

between open analytic curves; it corresponds to the finite ring extension $\mathcal{O}^\wedge_{Y_R,y}/\mathcal{O}^\wedge_{X_R,x}$ induced by the covering map Φ_R by completion in the respective points. It follows that ϕ is a finite covering map in the sense of Def. 1.17. Since Φ is a G-Galois cover, G acts on $\bar{Y}$. The stabilizer subgroup at $y \in \bar{Y}$ makes ϕ a $\mathrm{Stab}_G(y)$-Galois cover of open analytic curves.

We come back to the situation from Sect. 1.4.1, where we began to study the semistable reduction of the curve Y from a relative point of view by choosing a finite separable map to the projective line X. Normalization of the standard smooth model X_R led to the model Y_R and a covering map $\Phi_R : Y_R \to X_R$ (with X_R and Y_R assumed to be permanent). As before, we assume the special fiber of Y_R to contain a finite number of non-nodal singularities $y_1, \dots, y_n \in \bar{Y}$; their image points $x_i := \bar{\Phi}(y_i) \in \bar{X}$ are called *critical points*. Since the latter are smooth points of $\bar{X}$, their residue classes $\mathsf{X}_i :=]x_i[_{X_R}$ are open disks, whereas the residue classes $\mathsf{Y}_i :=]y_i[_{Y_R}$ of the former points are neither open disks nor open annuli, as follows from Lem. 1.13. The maps $\phi_i : \mathsf{Y}_i \to \mathsf{X}_i$ on residue classes give rise to the setting that we are mainly concerned with in this paper; namely, the study of non-trivial Galois covers of the rigid analytic open unit disk.

Denote by $\mathscr{Y}_i$ and $\mathscr{X}_i$ the minimal formal models corresponding to Y_i and X_i, respectively. As we will see in the next section, modifications in this local setting—that is, formal blow-ups of $\mathscr{Y}_i \to \mathscr{X}_i$—induce compatible modifications of the R-models we started with—that is, blow-ups of the covering $Y_R \to X_R$.

1.4.4. *Modifications Determined by Local Data.* In this section, X shall denote the rigid analytic open unit disk, realized as the residue class $]x_0[_{X_R}$ of a permanent model X_R. As usual, we let $A := \mathring{\mathscr{O}}_{\mathsf{X}}$ be the ring of zero-bounded analytic functions on X; by Lem. 1.13, A is a ring of formal power series, giving the canonical model $\mathscr{X} := \mathrm{Spf} A$ of the disk.

Definition 1.19. Let A be a power series ring over R. A *parameter for A* is a function $t \in A$ such that $A = R[\![t]\!]$. When A is considered as the ring of zero-bounded analytic functions on an open disk X, we also speak of a *parameter for X*.

Remark 1.20. The choice of a parameter for the open disk X defines an isomorphism $\mathsf{X} \cong \{x \in \mathbb{A}^1_K \mid 0 < v(t(x))\}$ and, in particular, fixes a center for the disk (though different parameters can determine the same disk center); see the proof of Lem. 1.28.

We also have the notion of closed subdisks.

Definition 1.21. A subset $\mathsf{D} \subset \mathsf{X}$ is called a *closed disk* if—after a finite extension of the base field K—there is a parameter t for the open disk X and a positive number $\rho \in \mathbb{R}_{>0}$ such that D is of the form

$$\mathsf{D} = \{x \in \mathsf{X} \mid \rho \le v(t(x))\} .$$

If D is also an affinoid subdomain of X (which happens precisely if $\rho \in v(K^{\mathrm{ac}\times})$), we call D an *affinoid disk*.

Remark 1.22. Note that closed disks need to have a representation as in the formula of Def. 1.21 only after a suitable finite extension of the base field. As long as one is dealing with a finite number of disks, this has little practical impact—one just works with a large enough finite extension over which all disks can be defined. Things get more complicated when infinitely many disks are involved; cf. the situation of Thm. 1.24.

An important observation is the fact that we can state modifications of the considered models in geometric terms. More precisely, affinoid subdisks of X correspond to formal blow-ups of $\mathscr{X}$, which in turn correspond to blow-ups of X_R with center $x_0 \in \bar{X}$. We explain the correspondence.

Let $\mathsf{D} \subset \mathsf{X}$ be an affinoid subdisk. Replacing K by a finite extension (if necessary), there exists a parameter $t \in A$ for X and a radius $\rho \in v(K^\times)$, with $\rho > 0$, such that $\mathsf{D} = \{x \in \mathsf{X} \mid \rho \le v(t(x))\}$. Denote by $a \in K$ any element with valuation $v(a) = \rho$. Via the ideal $(t, a) \lhd A$, the affinoid disk D gives rise to a formal blow-up $\mathscr{X}' \to \mathscr{X}$. The general fact that formal blow-ups are induced by algebraic blow-ups is easy to see in the current situation: We can assume the parameter $t \in A$ to be an element of the local ring $\mathscr{O}_{X_R, x_0}$ because every $t' \in \mathscr{O}_{X_R, x_0}$ sufficiently close to t is a parameter for X

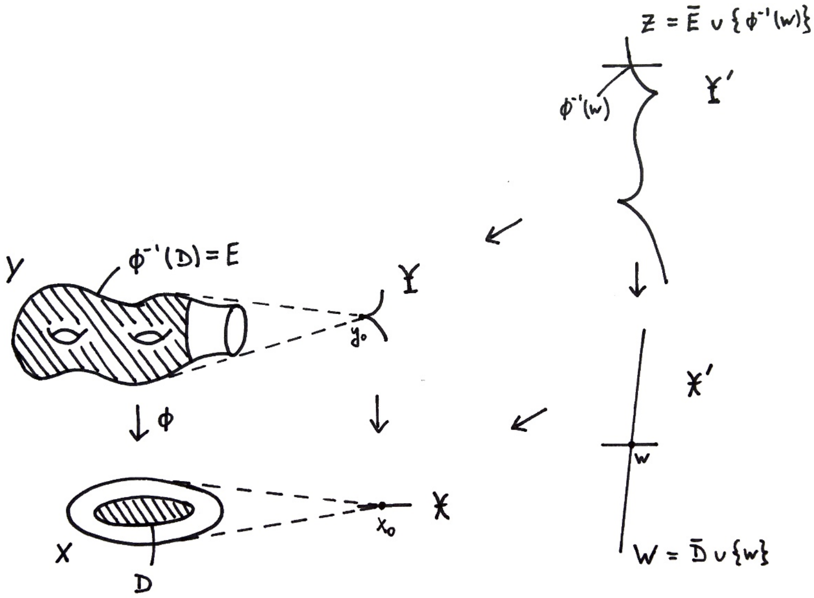

FIGURE 1.2. The open analytic curve Y is the residue class of the singular point $y_0 \in Y_R$; it is the generic fiber associated to the formal model $\mathscr{Y} = Y_R|^{\wedge}_{y_0}$. A covering $\mathscr{Y} \to \mathscr{X}$ of formal models induces a covering $\phi : \mathsf{Y} \to \mathsf{X}$ of rigid analytic curves; when $\mathscr{X} = X_R|^{\wedge}_{x_0}$ is formally smooth, $\mathsf{X} =]x_0[$ is an open unit disk. Modifying with respect to the minimal exhausting disk $\mathsf{D} \subset \mathsf{X}$ improves the situation—in the depicted case by splitting the singularity of $\mathscr{Y}$ into several singularities less bad.

determining the same ideal $(t', a) = (t, a) \lhd A$. The algebraic blow-up $f : X'_R \to X_R$ given by $(t, a) \lhd \mathscr{O}_{X_R,x}$ obviously induces the aforementioned formal blow-up $\mathscr{X}' \to \mathscr{X}$. It follows that $\mathscr{X}' = X'_R|^{\wedge}_W$ is the formal completion of X'_R along the exceptional divisor $W := f^{-1}(x_0)$, see [BL93, Sect. 2]. As explained in [AW11, Sect. 3.5], W is a rational component intersecting the strict transform of $\bar{X}$ in a unique point $w \in W$, which is an ordinary double point; in other words, f is a simple modification. Furthermore, the closed disk D can be recovered as the formal fiber of $W^\circ := W \setminus \{w\}$ (with respect to the model X'_R).

Now assume that we have a finite covering $\phi : \mathsf{Y} \to \mathsf{X}$ of open analytic curves, coming from a covering $\Phi_R : Y_R \to X_R$ of permanent R-models. As in Sect. 1.4.1, the modification f of X_R induces by normalization a modification $g : Y'_R \to Y_R$, which we can assume to be permanent (after a finite extension of the base field, if necessary; cf. Conv. 1.6). We call g the modification induced by D. Denoting by Z the exceptional divisor of g, we get a finite map $Z \to W$; the inverse image of w consists of those points of Z that also lie on the strict transform of $\bar{Y}$. It follows that $\mathsf{E} := \phi^{-1}(\mathsf{D})$ is the formal fiber of $Z^\circ := Z \setminus \bar{\Phi}'^{-1}(w)$. The effect of modifying the models Y_R and X_R is thus described by restricting the rigid analytic covering ϕ to the affinoid disk $\mathsf{D} \subset \mathsf{X}$ and its preimage $\mathsf{E} \subset \mathsf{Y}$, that is, by studying $\phi|_{\mathsf{E}} : \mathsf{E} \to \mathsf{D}$. The situation is illustrated in Fig. 1.2.

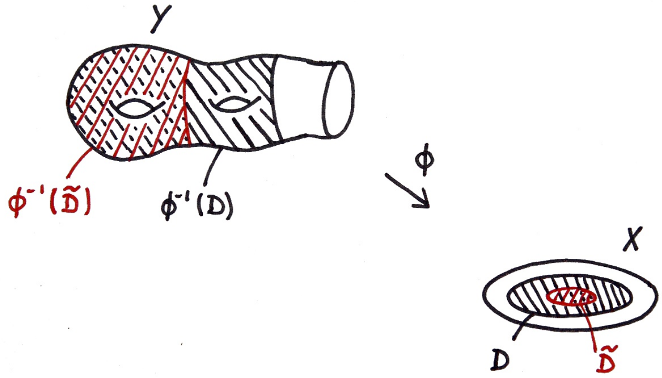

FIGURE 1.3. The affinoid disk $\mathsf{D} \subset \mathsf{X}$ is chosen large enough to be ϕ-exhausting: the complement of its preimage is an open annulus. By contrast, the preimage of a disk $\tilde{\mathsf{D}} \subset \mathsf{X}$ too small does not capture all genus of Y.

1.4.5. *Characterization of Improvements.* We get back to the setting and notation from Sect. 1.4.3. For the formal models $\mathscr{X}_i$ coming from the critical points $x_i \in X_R$, we would like to find suitable local modifications that induce improving modifications of the model Y_R. If we were able to determine these modifications, we would not only prove the relative version Thm. 1.15 of the Semistable Reduction Theorem, but with the arguments from Sect. 1.2.4, we would also have a practical resolution algorithm at hand. Indeed, if the improved model Y_R' were not yet semistable, we would start over again and examine the remaining critical points of X_R' (which by [AW11, Sect. 2.7] would be smooth points of $\bar{X}'$) and the corresponding rigid analytic coverings of open unit disks. As each iteration improves the situation in the sense of Sect. 1.2.3, finitely many of these improvement steps suffice to produce a semistable model for Y.

The crucial point is that the improving modifications we are looking for have a nice geometric interpretation: we will see with Thm. 1.24 and Prop. 1.25 below that the modification improving the situation at $y_i \in Y_R$ corresponds to the smallest affinoid subdisk $\mathsf{D}_i \subset \mathsf{X}_i$ that still captures 'enough information of the covering' $\mathsf{Y}_i \to \mathsf{X}_i$ in the sense of the next definition.

Definition 1.23. Let $\phi : \mathsf{Y} \to \mathsf{X}$ be a finite Galois cover of the rigid analytic open unit disk. An affinoid disk $\mathsf{D} \subset \mathsf{X}$ is called *exhausting* (with respect to ϕ) if the complement of its preimage $\mathsf{Y} \setminus \phi^{-1}(\mathsf{D})$ decomposes into a disjoint union of open annuli.

Figure 1.3 illustrates the intuition behind this notion: exhausting disks are large enough to capture all genus of the covering. By [BL85, Lem. 2.4], every sufficiently large closed disk $\mathsf{D} \subset \mathsf{X}$ is exhausting; a much more delicate task is to prove that there is a minimal exhausting disk.

Theorem 1.24. *Let $\phi : Y \to X$ be a finite Galois cover of the open unit disk. If Y is not a disk (that is, if the canonical model of Y is not formally smooth), then the set of ϕ-exhausting disks has a unique minimum with respect to inclusion.*

This theorem can be considered as a local version of the Semistable Reduction Theorem, in that Thm. 1.15 is a consequence of Thm. 1.24 and the following key result Prop. 1.25 (which is shown in [AW11, Lem. 3.10]); observe that the condition 'Y is not a disk' in the above theorem is satisfied for the coverings coming from the examination of critical points as described in Sect. 1.4.3.

Proposition 1.25. *Let the notation be as before and assume that $x_0 \in \bar{X}$ is a critical point of the covering $\Phi_R : Y_R \to X_R$. Then the modification $g : Y_R' \to Y_R$ induced by the affinoid disk $D \subset X =]x_0[_{X_R}$ is simple if and only if D is ϕ-exhausting; the modification is an improvement at every point $y \in \bar{\Phi}^{-1}(x_0)$ if and only if D is minimal among the set of ϕ-exhausting disks.* □

If we could prove Thm. 1.24 by explicitly constructing a minimal element—the so-called *minimal exhausting disk*—the algorithmic procedure explained above would allow us to actually compute the semistable reduction of any given curve. We will show in Sects. 2 and 4 that such an explicit construction is possible in certain special cases (to be introduced in the next section 1.4.6).

1.4.6. *Two Main Cases.* In [AW11], the local version Thm. 1.24 of the Semistable Reduction Theorem is proven except for two base cases, in which the existence of a minimal element has to be established by direct arguments. The purpose of the present dissertation is precisely to fill this gap by providing in both of these cases an explicit construction of the minimal element, thereby also completing our algorithmic proof of the Semistable Reduction Theorem.

More precisely, when the covering has a solvable Galois group, Thm. 1.24 can be reduced via an induction argument to the case where the covering is cyclic of prime order. In this special situation, we can explicitly construct the minimal exhausting disk, as will be shown in Sect. 2 of this paper. For the induction step to work, however, it does not suffice to only study coverings of the open unit disk but prime-cyclic covers of open annuli also have to be considered. In the latter case, we have to establish the existence of the so-called maximal separating boundary domain, a generalization of the notion of minimal exhausting disks. Again, we can provide an explicit construction of the maximal element; see Sect. 4.

When the Galois group is not solvable, one faces additional problems in that it is more difficult to rule out the possibility that the intersection of the set of exhausting disks is empty. This obstacle can be overcome by considering the Berkovich analytic space associated to the open unit disk. The situation can then be simplified by reducing to the solvable case through considering certain inertia subgroups of the original Galois group, which are solvable again; see [AW11, Sect. 5] for the details. We expect that this reduction step can also be made explicit.

1.5. **Separating Boundary Domains.** To be able to establish the existence of a minimal exhausting disk, we will need some criteria to recognize when a given closed disk is exhausting and when it is minimal with this property. As we will also have to deal with covers of open annuli, we have to slightly generalize the notion of exhausting disks.

1.5.1. *Definition.* Let X be the rigid analytic open unit disk or an open annulus, and let $A := \mathring{\mathcal{O}}_{\mathsf{X}}$ denote the ring of zero-bounded analytic functions on X. Let $\eta \in \partial\mathsf{X}$ be a boundary point of X. If X is a disk, there is no choice; if X is an annulus, this amounts to choosing an 'orientation' of X. A *parameter for X with respect to η* is an element $t \in A$ with $v_\eta(t) = (0,1)$. If X is a disk, then $A = R[\![t]\!]$ and we get an isomorphism

$$X \cong \{x \in \mathbb{A}_K^1 \mid 0 < v(t(x))\} ;$$

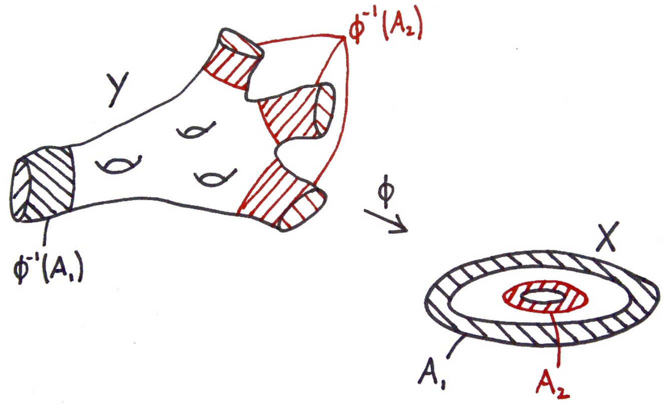

FIGURE 1.4. The notion of separating boundary domains generalizes the notion of exhausting disks and is applicable in situations where coverings of open annuli are studied. In the picture, one annulus lies above the 'outer' boundary domain $\mathsf{A}_1 \subset \mathsf{X}$ and three annuli lie above the 'inner' boundary domain $\mathsf{A}_2 \subset \mathsf{X}$; both A_1 and A_2 are thus ϕ-separating.

see the proof of Lem. 1.28. In this case, the new terminology agrees with the old one from Def. 1.19. If X is an open annulus, there exists $a \in K$ with $v(a) > 0$ such that $s := a/t \in A$ is a parameter for X with respect to the boundary point distinct from η, and the choice of t yields an isomorphism

$$\mathsf{X} \cong \{x \in \mathbb{A}^1_K \mid 0 < v(t(x)) < \epsilon\}\,,$$

where $\epsilon := v(a)$; see the proof of Lem. 1.30.

Definition 1.26. Let X be an open disk or an open annulus, and denote by $A := \mathring{\mathcal{O}}_{\mathsf{X}}$ the ring of analytic functions on X bounded by 0. Let $\eta \in \partial\mathsf{X}$ be given. A *boundary domain* of X with respect to η is a rigid analytic open subspace $\mathsf{A} \subset \mathsf{X}$ of the form

$$\mathsf{A} = \{x \in \mathsf{X} \mid v(t(x)) < \epsilon\}\,,$$

where $t \in A$ is a parameter for X with respect to η and where $\epsilon > 0$ is an element of the value group $v(K^\times)$. The boundary domain A is called *separating* for a finite Galois covering $\phi : \mathsf{Y} \to \mathsf{X}$ if $\phi^{-1}(\mathsf{A})$ decomposes into a disjoint union of open annuli; see the illustration in Fig. 1.4.

Remark 1.27. Let X be an open disk and let $\phi : \mathsf{Y} \to \mathsf{X}$ be a finite Galois covering. Then $\mathsf{A} \subset \mathsf{X}$ is a (maximal) ϕ-separating boundary domain if and only if $\mathsf{X} \setminus \mathsf{A}$ is a (minimal) ϕ-exhausting disk.

1.5.2. *Recognizing Separating Boundary Domains.* In order to be able to recognize exhausting disks (or more generally, separating boundary domains), it is essential that we can identify open annuli. We start with a lemma that allows us to detect open disks.

Lemma 1.28. *Let X be an open analytic curve having exactly one boundary point $\eta \in \partial X$. Denote by $A := \mathring{\mathcal{O}}_X$ the ring of analytic functions on X that are bounded by* 0. *Assume, there is $t \in A$ satisfying $v_\eta(t) = (0,1)$. Then X is isomorphic to*

$$\{x \in \mathbb{A}^1_K \mid 0 < v(t(x))\}\,;$$

that is, X is an open disk and t is a parameter for X.

For the proof, we will need the following well-known fact from commutative algebra (which is an easy consequence of [AM69, Lem. 10.23]).

Lemma 1.29. *Let $\psi : (S, \mathfrak{m}_S) \to (T, \mathfrak{m}_T)$ be a local homomorphism of complete Noetherian local rings. If the induced maps $S/\mathfrak{m}_S \to T/\mathfrak{m}_T$ and $\mathfrak{m}_S/\mathfrak{m}_S^2 \to \mathfrak{m}_T/\mathfrak{m}_T^2$ are isomorphisms, then ψ is also an isomorphism.* □

Proof of Lem. 1.28. Let $\bar{A} := A/(\pi)$ denote the ring corresponding to the canonical reduction of X. The open analytic curve X comes by definition from a permanent model, so $\bar{A}$ is reduced; it is even an integral domain, as X has by assumption only one boundary point. We conclude that $\eta \in \partial\mathsf{X}$ corresponds to the height-one prime ideal $(\pi) \lhd A$ and that $\#_{(\pi)}$ (the discrete valuation, which together with $v_{(\pi)}$ constitutes the rank-two-valuation v_η) is defined on $\mathrm{Frac}\,\bar{A}$. The assumptions on t imply that its image $\bar{t} \in \bar{A}$ is a uniformizer for the discrete valuation on $\mathrm{Frac}\,\bar{A}$; the complete local ring $\bar{A}$ thus coincides with the valuation ring $k[\![\bar{t}]\!]$ of $\#_{(\pi)}$. As a consequence, the maximal ideal of A is $\mathfrak{m}_A = (\pi, t)$.

With the assumptions of Lem. 1.29 being satisfied for the local homomorphism $R[\![t]\!] \to A$, it follows that $R[\![t]\!]$ and A are isomorphic; that is, X is an open disk with parameter t. □

The next lemma is similar to Lem. 1.28; it lets us detect open annuli.

Lemma 1.30. *Let Y be an open analytic curve having exactly two boundary points $\xi_1, \xi_2 \in \partial\mathsf{Y}$. Denote by $B := \mathring{\mathcal{O}}_{\mathsf{Y}}$ the ring of zero-bounded analytic functions on Y, and set $B_K := B \otimes K$. Assume, there is $t \in B_K^\times$ satisfying*

$$v_{\xi_1}(t) = (0, 1) \quad \text{and} \quad v_{\xi_2}(t) = (\epsilon, -1)$$

for some $\epsilon > 0$ in the value group $v(K^\times)$. Then Y is isomorphic to

$$\{y \in \mathbb{A}_K^1 \mid 0 < v(t(y)) < \epsilon\}\,;$$

that is, Y is an open annulus of thickness ϵ and t is a parameter for Y.

Proof. Let $\mathfrak{q}_1 \lhd B$ and $\mathfrak{q}_2 \lhd B$ be the two minimal prime ideals of B lying over $(\pi) \lhd R$; they correspond to the two boundary points $\xi_1 \in \partial\mathsf{Y}$ and $\xi_2 \in \partial\mathsf{Y}$. Due to assumption, t lies in both valuation rings $B_{\mathfrak{q}_1}$ and $B_{\mathfrak{q}_2}$. Therefore,

$$t \in B_K \cap B_{\mathfrak{q}_1} \cap B_{\mathfrak{q}_2} = B\,.$$

The last equality holds true because a normal integral domain is the intersection of the localizations at all its prime ideals of height-one, see [Mat89, Thm. 11.5].

The assumptions on t also imply that the image $\bar{t} \in B/\mathfrak{q}_1$ is a uniformizer with respect to the discrete valuation $\#_{\mathfrak{q}_1}$ on $\mathrm{Frac}\,B/\mathfrak{q}_1$, so that $B/\mathfrak{q}_1 = k[\![\bar{t}]\!]$. Denoting by $a \in R$ an arbitrary element with $v(a) = \epsilon$, the function $s := a/t$ satisfies the same assumptions as t does, only with the roles of ξ_1 and ξ_2 interchanged. Consequently, $s \in B_K^\times$ is an element of B, and $B/\mathfrak{q}_2 = k[\![\bar{s}]\!]$.

Taking both results together yields $B/(\pi) = k[\![\bar{s}, \bar{t} \mid \bar{s}\bar{t} = 0]\!]$. Indeed, we have a sequence

$$0 \longrightarrow B/(\pi) \longrightarrow B/\mathfrak{q}_1 \oplus B/\mathfrak{q}_2 \longrightarrow k \longrightarrow 0\,,$$

where the first map sends the residue class $\bar{f} \in B/(\pi)$ to the pair $(f \bmod \mathfrak{q}_1, f \bmod \mathfrak{q}_2)$, and where the second map sends $(f_1 \bmod \mathfrak{q}_1, f_2 \bmod \mathfrak{q}_2)$ to the residue class of $f_1 - f_2$

in $B/\mathfrak{m}_B = k$ (here $\mathfrak{m}_B \lhd B$ denotes the maximal ideal of the local ring B). This sequence is seen to be exact: The injectivity of the first map follows from the equality $(\pi) = \mathfrak{q}_1 \cap \mathfrak{q}_2$, which can be deduced using [Mat89, Thm. 11.5] once more. Exactness in the middle is equivalent to $\mathfrak{q}_1 + \mathfrak{q}_2 = \mathfrak{m}_B$, which holds true because

$$B/(\mathfrak{q}_1 + \mathfrak{q}_2) = (B/\mathfrak{q}_1)/\bar{\mathfrak{q}}_2 = k[\![\bar{t}]\!]/(\bar{t}) = k\,;$$

for this, note that $t \in B$ is an element of $\mathfrak{q}_2$, since $v_{\mathfrak{q}_2}(t) = \epsilon > 0$. The surjectivity of the second map is clear. As the kernel of the second map is $k[\![\bar{s}, \bar{t} \mid \bar{s}\bar{t} = 0]\!]$, the asserted equality with $B/(\pi)$ follows from the exactness of the sequence.

With [Liu02, Lem. 10.3.20], we deduce that $B = R[\![s, t \mid st = a]\!]$, which implies that Y is an open annulus of the asserted form and with t a parameter with respect to $\xi_1 \in \partial\mathsf{Y}$. □

When considering covers $\mathsf{Y} \to \mathsf{X}$ of open analytic curves, with X an open annulus, we can use the above lemma to deduce under certain conditions (which are quite easy to check in practice) that Y is also an open annulus. In other words, with Lem. 1.31, we obtain a criterion that allows us to recognize separating boundary domains; the criterion will be applied, for example, in the proof of Prop. 2.33.

Lemma 1.31. *Let X be an open annulus of thickness ϵ and let $\mathsf{Y} \to \mathsf{X}$ be a finite covering of open analytic curves of degree n. Denote by $B := \mathring{\mathcal{O}}_{\mathsf{Y}}$ and $A := \mathring{\mathcal{O}}_{\mathsf{X}}$ the corresponding rings of zero-bounded analytic functions. Let $t \in A$ be a parameter for X. Assume that Y*[4] *has exactly two boundary points $\xi_1, \xi_2 \in \partial\mathsf{Y}$ and that there is $w \in B$ satisfying*

(1) $\mathrm{Norm}_{B/A}(w) = t^m u$, *with a unit* $u \in A^\times$,
(2) $\gcd(n, m) = 1$.

Then Y is an open annulus of thickness ϵ/n.

Proof. Due to assumption (1), the norm of w is a unit in $A_K := A \otimes K$. By the same argument as in the previous lemma 1.30, it follows that w is a unit of B_K.

The two boundary points $\xi_1, \xi_2 \in \partial\mathsf{Y}$ lie above the two boundary points $\eta_1, \eta_2 \in \partial\mathsf{X}$, respectively (cf. Rem. 1.18). For $i \in \{1, 2\}$, the corresponding extension of rank-two-valuations v_{ξ_i}/v_{η_i} is weakly unramified with respect to the first component and totally ramified of degree n with respect to the second component. Indeed, denote by $\mathfrak{q}_i \lhd B$ and $\mathfrak{p}_i \lhd A$ the height-one prime ideals corresponding to ξ_i and η_i. Because the open analytic curves Y and X are obtained as residue classes of permanent models, the extension of discrete valuation rings $B_{\mathfrak{q}_i}/A_{\mathfrak{p}_i}$ is weakly unramified, as both value groups coincide with the value group of the valuation on K; see Sect. 1.3.5. As ξ_i is the only boundary point of Y lying above $\eta_i \in \partial\mathsf{X}$, we deduce from the fundamental equality relating ramification and inertia degree with the extension degree that the extension of residue fields is of degree $[\mathrm{Frac}\, B/\mathfrak{q}_i : \mathrm{Frac}\, A/\mathfrak{p}_i] = n$. Both residue fields are complete discretely valued fields, and these, in turn, have the algebraically closed k as their residue field; consequently, the extension has to be totally ramified of degree n, as asserted.

Using assumption (1), we can thus calculate

$$v_{\xi_1}(w) = v_{\xi_1}(t^m u)/n = (0, m) \quad \text{and} \quad v_{\xi_2}(w) = v_{\xi_2}(t^m u)/n = (m\epsilon/n, -m)\,.$$

Due to assumption (2), there exist $a, b \in \mathbb{Z}$ with $am + bn = 1$. Setting

$$v := w^a t^b \in B_K^\times\,, \tag{1.2}$$

[4] Recall that the open analytic curve Y is connected; see Rem. 1.8.

it is easy to verify that v satisfies the assumptions of Lem. 1.30 with regard to Y. As a consequence, Y is an open annulus of thickness ϵ/n and v is a parameter with respect to $\xi_1 \in \partial$Y. □

1.5.3. *Recognizing Improvements.* In the previous section, we have deduced the criterion Lem. 1.31 to recognize separating boundary domains. With respect to Prop. 1.25, we will also need to know when a given exhausting disk is minimal (that is, when the corresponding modification is an improvement), or more generally, when a given separating boundary domain is maximal.

The following facts come from the considerations in [AW11, Lem. 2.9]. The first lemma is well-suited for the study of disk coverings as carried out in Sect. 2, while the second lemma is better suited for our study of coverings of open annuli in Sect. 4.

Lemma 1.32. *With the usual notation, let $\phi: Y \to X$ be a covering of the rigid analytic open unit disk induced by a finite Galois covering $\Phi_R: Y_R \to X_R$ of permanent R-models through localizing in a smooth critical point of X_R. The modification associated to the exhausting disk $D \subset X$ is an improvement of the model Y_R, if and only if the reduction $\bar{E}$ of the affinoid $E := \phi^{-1}(D)$ is either smooth of positive genus or contains at least two singular points.* □

Lemma 1.33. *With the usual notation, let $\phi: Y \to X$ be a covering of a rigid analytic open annulus X of thickness ϵ, induced by a finite Galois covering $\Phi_R: Y_R \to X_R$ of permanent R-models through localizing in an ordinary double point of X_R. Let $t \in \mathring{\mathcal{O}}_X$ be a parameter of X with respect to the boundary point $\eta \in \partial X$, and assume the boundary domain $A = \{x \in X \mid v(t(x)) < \rho\}$, with $\rho \in v(K^\times)$ and $0 < \rho < \epsilon$, to be separating. Then A is maximal, if and only if the reduction of the affinoid*

$$\phi^{-1}(\{x \in X \mid \rho = v(t(x))\}) \subset Y$$

is either smooth or contains a singular point. □

2. Prime-Cyclic Étale Galois Covers of the Open Unit Disk

As mentioned in Sect. 1.4.6, one of our objectives is to determine the minimal exhausting disk for prime-cyclic Galois coverings of the open unit disk. As explained there, this case serves as the induction basis when dealing with a general solvable Galois group; we will deal with this situation in this section, but under the additional assumption of the covering being étale. In Sect. 4, we will handle the ramified case and, somewhat more generally, study coverings of open annuli (the other situation that has be taken care of for the induction argument to work).

2.1. **Setting.** Throughout this section, we assume K to be of mixed characteristic $(0,p)$. We will study a prime-cyclic étale Galois covering ϕ : Y → X of the rigid analytic open unit disk, and we want to prove Thm. 1.24 in the case that the covering is of degree p. This really is the essential and hardest case, and it is the contents of the next theorem (the proof of which will occupy the rest of this section).

Theorem 2.1. *Let $\phi: Y \to X$ be a p-cyclic étale Galois cover of the rigid analytic open unit disk X. If the covering is non-trivial (that is, if Y is not a disk) then the set of ϕ-exhausting disks has a unique minimum with respect to inclusion, and this minimum can explicitly be constructed.*

As before, we denote by $B := \mathring{\mathcal{O}}_{\mathsf{Y}}$ and $A := \mathring{\mathcal{O}}_{\mathsf{X}}$ the corresponding rings of zero-bounded analytic functions, giving the canonical models $\mathscr{Y} := \operatorname{Spf} B$ and $\mathscr{X} := \operatorname{Spf} A$. The assumption that ϕ is a p-cyclic Galois cover translates by Def. 1.17 into the following: B is the integral closure of A in the p-cyclic Galois extension $\operatorname{Frac} B / \operatorname{Frac} A$. By Kummer theory, the latter is given by an irreducible equation of the form

$$y^p = f \ , \tag{2.1}$$

with an element $f \in (\operatorname{Frac} A)^\times$ that is not a pth power in $\operatorname{Frac} A$. This requires K to be large enough to contain a primitive pth root of unity; by Conv. 1.6, we may (and will) assume K to do so.

Multiplying with appropriate pth powers, we can assume f to be an element of A. Choosing a parameter $t \in A$ (therewith also fixing a center for the disk X), we can write $A = R[\![t]\!]$ and express f as a power series in t over R,

$$f = \sum_{i=0}^{\infty} a_i t^i \ .$$

Due to our assumption that the covering shall be étale, we can even assume that f is a unit of A. Indeed, by Weierstraß preparation (for example, [BGR84, Sect. 5.2.2]), we can write $f = bPu$, with an element $b \in R \setminus \{0\}$, a distinguished Weierstraß polynomial $P \in R[t]$ of degree n, and an invertible power series $u \in R[\![t]\!]^\times$. Replacing K by a finite extension (if necessary), we can replace y by $b^{1/p} y$ and assume $b = 1$. Then, if f is not a unit of A, there has to be $x \in K$ with $f(x) = P(x) = 0$ (again, it might be necessary to pass to a finite extension of K). We have $v(x) > 0$, as can be deduced from the Newton polygon of P (see Sect. 2.2.3 for a quick reminder of this concept). The zero $x \in R$ therefore corresponds to a point of the open unit disk X. But then the multiplicity of x in P has to be divisible by p—otherwise the covering would be ramified in this point, contradicting our assumption. After a variable change collecting all powers $(t - x)^p$ into y, we can hence assume that $P = 1$ and, accordingly, that $f \in A^\times$ is a unit.

In the following, we will often refer to the above setting, which we call the *geometric situation*: that is,

$$\text{the power series } f \in A \text{ is a unit of } A \text{ and not a } p\text{th power in } A\,. \tag{GEOM}$$

Recall that a formal power series is invertible precisely when its constant coefficient is a unit (so after a further extension of K, we could assume $a_0 = 1$).

2.2. **Special Fiber.** In order to determine appropriate modifications of $\mathscr{Y}$, we first have to understand what the special fiber of $\mathscr{Y}$ looks like. Just reducing the cover defining equation $y^p = f$ is often of limited use since y might not generate B over A. Or, the reduced equation might describe a non-reduced scheme, whereas by Conv. 1.6, the models we are studying are all assumed to be permanent. As an example, consider the 3-cyclic Galois cover given by the equation $y^3 = 1 + 3t^3 + 3t^5$: reduction modulo 3 leads to the non-reduced scheme given by the third power $\bar{y}^3 = 1$.

The basic idea is to first approximate f by a pth power h^p 'as good as possible' and then make a change of coordinates (heuristically speaking, all pth powers contained in f should be taken care of by y^p). This will lead to an equation for a generator w of B over A, and this equation will be suitable for reduction.

2.2.1. *Best Approximation.* The next proposition shows how to obtain an algebraic description of the covering $\mathsf{Y} \to \mathsf{X}$, that is, of the corresponding ring extension B/A of zero-bounded analytic functions. Similar assertions in the context of discrete valuation rings are well-known, cf. [Hen00].

Proposition 2.2. *Let $Y \to X$ be a p-cyclic étale Galois cover of the rigid analytic open unit disk, and denote by $B := \mathring{\mathcal{O}}_{\mathsf{Y}}$ and $A := \mathring{\mathcal{O}}_{\mathsf{X}}$ the corresponding rings of zero-bounded analytic functions. Assume that the covering is given by a Kummer equation $y^p = f$, with $f \in A^\times$ as in (GEOM). Denote by $\eta \in \partial X$ the unique boundary point of the disk and by v_η the associated rank-two-valuation. Then the following holds true:*

(1) *The subset*

$$\{v_\eta(f - h^p) \mid h \in A\} \subset \mathbb{Q} \times \mathbb{Z}$$

has a unique maximum (μ, m), with $0 \leq \mu < p/(p-1)$ and with m prime to p.

(2) *Let $h \in A$ be any best-approximating element (that is, an element giving the maximal possible value $v_\eta(f - h^p) = (\mu, m)$).*
   (a) *If $\mu = 0$, we have $B = A[w]$, with $w \in B$ satisfying $w^p = f$.*
   (b) *Otherwise, we have $B = A[w]$, with $w := (y - h)/\lambda \in B$ satisfying*

$$\frac{(\lambda w + h)^p - h^p}{\lambda^p} = w^p + \cdots + p\lambda^{1-p}h^{p-1}w = \frac{f - h^p}{\lambda^p}$$

(and where $\lambda \in K$ is some element of valuation $v(\lambda) = \mu/p$).

Remark 2.3. The classical notion of Artin conductors (as in [Ser79, Sect. VI.2]) can be generalized by defining them in terms of the rank-two-valuations coming from the boundary points of open analytic curves. For example, in the situation of the corollary, the Artin conductor at $\eta \in \partial\mathsf{X}$ is $a_\eta = (p-1)(1-m)$. These generalized Artin conductors measure the ramification at the boundaries of open analytic curves and can be used to formulate another criterion to recognize separating boundary domains. But since our intention is to keep all calculations as simple and practical as possible, we stick to the more basic criteria explained in Sect. 1.5.2.

Proof of Prop. 2.2. We first assume that $\bar{f} \in \bar{A} := A/(\pi) = k[\![\bar{t}]\!]$ is not a pth power; then the Kummer equation $y^p = f$ remains irreducible over $\bar{A}$, and for all $h \in A$, we have $v_{(\pi)}(f - h^p) = 0$. Hence, in this case, $\mu = 0$ is the maximal value that can occur. Also, the maximal value for $m = \#_\eta(f - h^p)$ is seen to be $\mathrm{ord}_{\bar{t}}(\mathrm{d}\bar{f}) + 1$, which is prime to p. This is because changing h changes $\overline{f - h^p}$ by a pth power of $k[\![\bar{t}]\!]$—these are precisely the power series in $\bar{t}^p$ over k—and every change by a pth power can be obtained in this way. We will show further below that $w \in B$ with

$$w^p = f \tag{2.2}$$

generates the ring extension B/A.

Next, we consider the case where $\bar{f} \in \bar{A}^p$. The proof is similar to the previous case, but it is more difficult to justify that the maximum of $v_{(\pi)}(f - h^p)$ is attained. We choose $h \in A^\times$ such that $\mu := v_{(\pi)}(f - h^p) > 0$. Set $\tilde{g} := f - h^p$ and consider the variable change $\tilde{w} := y - h$ leading to the irreducible equation

$$\tilde{F}(\tilde{w}) = (\tilde{w} + h)^p - h^p - \tilde{g} = \tilde{w}^p + \cdots + ph^{p-1}\tilde{w} - \tilde{g} = 0\,, \tag{2.3}$$

where $\tilde{F}$ is a monic irreducible polynomial of degree p over A. Choosing $\lambda \in K$ with $v(\lambda) \le \mu/p$ and setting $g := \tilde{g}/\lambda^p$, we can make a further variable change

$$w := \frac{\tilde{w}}{\lambda} = \frac{y-h}{\lambda}\,, \tag{2.4}$$

resulting in the irreducible polynomial equation

$$F(w) = \frac{(\lambda w + h)^p - h^p}{\lambda^p} - g = w^p + \cdots + ph^{p-1}\lambda^{1-p}w - g = 0\,. \tag{2.5}$$

We are going to show that we always have $\mu < p/(p-1)$. Indeed, by contradiction suppose that $\mu \ge p/(p-1)$. We can then make the variable change (2.4) choosing $\lambda := \zeta_p - 1$, with $\zeta_p \in K$ a primitive pth root of unity; note that $v(\lambda) = 1/(p-1) \le \mu/p$. Reducing Eq. (2.5) modulo the maximal ideal $(\pi, t) \lhd A$ results in an Artin–Schreier type equation over the algebraically closed residue field $A/(\pi, t) = k$, which has p distinct solutions (one has to use the fact that h has, as an invertible power series, a unit as leading coefficient). Since A is complete with respect to (π, t), those solutions over k lift by Hensel's lemma to solutions in A—contradicting the fact that F is irreducible. As a consequence, the possible values for $v_{(\pi)}(f - h^p)$, with $h \in A$, are bounded from above. Because $v_{(\pi)}$ is discrete, we can choose $h \in A^\times$ such that the maximal possible value $\mu < p/(p-1)$ is attained.

We still have to see that the rank-two-valuation v_η attains a maximum (μ, m). We first show that K contains an element with valuation μ/p. For this, we come back to Eq. (2.3) and consider an arbitrary (but fixed) extension $v_{\mathfrak{q}}$ of $v_{(\pi)}$ to $\operatorname{Frac} B$. Since B/A is finite, $v_{\mathfrak{q}}$ dominates B and thus corresponds to a height-one-prime ideal $\mathfrak{q} \lhd B$, with $\pi \in \mathfrak{q}$. By the same reasoning as before, $v_{\mathfrak{q}}/v_{(\pi)}$ is seen to be weakly unramified: open analytic curves are defined in terms of permanent models and the value groups of $v_{\mathfrak{q}}$ and $v_{(\pi)}$ therefore coincide with $v(K^\times)$; see Sect. 1.3.5. Then, by the strong triangle inequality, since $v_{\mathfrak{q}}(\tilde{g}) = v_{(\pi)}(\tilde{g}) = \mu < p/(p-1)$, at least one of the other terms in Eq. (2.3) needs to have valuation μ as well. One easily sees that we necessarily have $v_{\mathfrak{q}}(\tilde{w}) = \mu/p$. Consequently, there is also a field element of K having this valuation. We can thus consider the variable change (2.4) under the assumption that h gives the best possible approximation $\mu = v_{(\pi)}(f - h^p)$ and $\lambda \in K$ has valuation $v(\lambda) = \mu/p$. Then the reduction $\bar{F} := F \bmod (\pi)$ remains irreducible over $\bar{A} := A/(\pi)$, which is to say that $\bar{g}$ is not a pth power. Indeed, $\bar{F}$ is a purely inseparable equation of the form $\bar{w}^p = \bar{g}$. If we had $\bar{g} = \bar{q}^p$, we could set $h_1 := h + \lambda q$, with $q \in A$ denoting any lift of $\bar{q}$. But then $v_{(\pi)}(f - h_1^p) > v_{(\pi)}(f - h^p)$—contradicting the choice of h. As in the case $\mu = 0$, it is now easy to see that the second component of v_η also attains a maximum m when $h \in A$ runs over all elements giving the maximal value $\mu = v_{(\pi)}(f - h^p)$ for the first component, and that m is maximal if and only if $(m, p) = 1$.

To finish the proof of the proposition, it remains to show that B/A is generated by $w \in B$ satisfying Eq. (2.2) in the first case and satisfying Eq. (2.5) otherwise (in the latter case, w shall be as in (2.4), with $h \in A$ giving the best possible approximation[5] $(\mu, m) = v_\eta(f - h^p)$ and with $\lambda \in K$ of valuation $v(\lambda) = \mu/p$). For this, it suffices to see that $A[w]$ is normal. Indeed, this would allow us to identify $A[w]$ with the integral closure B of A in $\operatorname{Frac} B$, as $A[w]$ is integral over A and has $\operatorname{Frac} B$ as its fraction field. Since $A[w]$ is Cohen–Macaulay (as a complete intersection over the local Noetherian ring A), by Serre's criterion on normality it is normal if and only if it is normal at its

[5] For the argument to follow, it would actually suffice to choose any element $h \in A$ maximizing the first component $v_{(\pi)}$ of the rank-two-valuation v_η.

points of codimension one, see [Liu02, Thm. 8.2.23 and Cor. 8.2.24]. These points are either closed points of the generic fiber, which are normal because w generically describes the étale cover $\mathsf{Y} \to \mathsf{X}$ of rigid analytic curves, or generic points of the special fiber. There is precisely one of the latter points because the ideal (π) remains prime in $A[w]$: as seen above, $\bar{F} := F \bmod (\pi)$ remains irreducible over $\bar{A} := A/(\pi)$.

To show normality at $(\pi) \lhd A[w]$, we have to see that the corresponding localization $A[w]_{(\pi)} = S[w]$ is normal. Here we have written $S := A_{(\pi)}$ for the valuation ring of $v_{(\pi)}$; denote its integral closure in $\operatorname{Frac} B$ by T. Since $\bar{F}$ remains irreducible over $\bar{S} := S/(\pi) = \operatorname{Frac} A/(\pi)$, the fundamental equality relating ramification degree, inertia degree, and extension degree shows that T has to be a single discrete valuation ring as well (or in other words, that the valuation $v_{(\pi)}$ has only one extension to $\operatorname{Frac} B$, which also means that there is only one boundary point of Y lying above $\eta \in \partial \mathsf{X}$). Moreover, the extension of residue fields $\bar{T}/\bar{S}$ is generated by $\bar{w}$. Then Nakayama's lemma implies that w generates T over S; that is, $T = S[w]$. As a consequence, $S[w]$ is normal. All in all, we conclude that $A[w]$ is normal, therewith finishing the proof of the proposition. □

Corollary 2.4. *In the situation of Prop. 2.2, there is precisely one boundary point $\xi \in \partial Y$ lying above $\eta \in \partial X$, the unique boundary point of the disk.*

Proof. See the last section in the proof of Prop. 2.2. □

Corollary 2.5. *When the covering $Y \to X$ in the statement of Prop. 2.2 is non-trivial (that is, when Y is not a disk itself), we always have $m > 1$. In particular, this holds true in the situation of Thm. 2.1.*

Proof. Suppose that we have $m = 1$. Then the element $w \in B$ (notation as in the proof of Prop. 2.2) satisfies $v_\xi(w) = (0,1)$, where ξ denotes the unique boundary point of Y (note that the rank-two-valuation v_ξ/v_η is weakly unramified with respect to the first component and totally ramified of degree p with respect to the second component; cf. the proof of Lem. 1.31). But then, by Lem. 1.28, Y is seen to be a disk—contradiction. □

Remark 2.6. In principle, the procedure described in the proof of Prop. 2.2 allows to produce a best approximation of f by a pth power. However, this requires the base field K to be large enough to have v_ξ/v be weakly unramified. This is all right for our new proof of the Semistable Reduction Theorem because, in theory, we can always assume to work with permanent models. But in practice, we usually do not know how large the base field has to be; a practical way of determining best approximations therefore requires other means (for example, the method of formal p-Taylor expansions to be introduced in the next section 2.2.2). Nevertheless, the above algorithm will prove very useful in the situation Sect. 4.5.2 is devoted to.

It is easy to recognize best approximations.

Lemma 2.7. *Let the power series $f \in A^\times$ be as in the geometric situation (GEOM). Denote by (μ, m) the value of a best possible approximation of f by a pth power in the sense of Prop. 2.2. Let $\tilde{h} \in A$ be given and set $(\tilde{\mu}, \tilde{m}) := v_\eta(f - \tilde{h}^p)$. If $p \nmid \tilde{m}$ then $\tilde{h}^p$ provides a best possible approximation of f—that is, $(\tilde{\mu}, \tilde{m}) = (\mu, m)$.*

Proof. Let $h \in A^\times$ be any element giving the best possible approximation (μ, m) of f by a pth power. By contradiction, suppose that $v_\eta(f - \tilde{h}^p) < (\mu, m)$. Then the strong triangle inequality shows that

$$v_\eta(f - \tilde{h}^p) = v_\eta(\tilde{h}^p - h^p) < (\mu, m)\,. \tag{2.6}$$

Write $\tilde{h} = h + \Delta$ and set $(\alpha, k) := v_\eta(\Delta)$. Since h is a unit in A, we derive from the binomial expansion $\tilde{h}^p - h^p = ph^{p-1}\Delta + \cdots + \Delta^p$ the inequality

$$v_\eta(\tilde{h}^p - h^p) \geq \min\{(1+\alpha,\, k),\ (p\alpha, pk)\}\,, \tag{2.7}$$

with equality holding true in case $1 + \alpha \neq p\alpha$. Using (2.6) and (2.7), we conclude that $p/(p-1) > \mu \geq \min\{1+\alpha,\, p\alpha\}$, and therefore inequality (2.7) can be strengthened to

$$v_\eta(\tilde{h}^p - h^p) = (p\alpha, pk)\,.$$

Indeed, suppose that $1+\alpha \leq p\alpha$; then $p/(p-1) > 1+\alpha$ and consequently $1+\alpha > p\alpha$—contradiction. But then

$$v_\eta(f - \tilde{h}^p) = (p\alpha, pk)$$

follows, contradicting the assumption that $\tilde{m}$ is prime to p. □

Remark 2.8. In the above proof, we do not use the fact that K is discretely valued—we only use the strong triangle inequality (which holds true for all algebraic extensions of K) to compare a best possible approximation with another approximation. We conclude that best possible approximations remain so under base change to arbitrary algebraic extensions of K, and that best approximations can be recognized with the condition of the lemma (that is, $\#_\eta(f - h^p)$ prime to p). This becomes important in the proof of Prop. 2.31: there, we have to deal with approximating power series defined over the integral extension R^{ac}/R of infinite degree and we need the uniqueness result from Prop. 2.28, which is based on the above Lem. 2.7.

Our next task is to actually determine an element h such that h^p approximates f as good as possible. This is done by extending a method introduced by Matignon, see below.

2.2.2. *Formal p-Taylor Expansion.* In earlier work, Matignon and Lehr also have examined p-cyclic Galois covers of the projective line, but they did so under the assumption of equidistant geometry. The latter means that for the projective line, there exists a smooth model separating the branch points of the covering. In this specific situation, the semistable model of the considered curve is of a rather simple structure; namely, the reduction is treelike. This allows to make ad-hoc assertions on where to find the separable components. Lehr [Leh01] started by offering an algorithm that gives the stable model under the additional condition that the cardinality of the branch locus is not greater than p. In this case, the necessary modifications can be read off from the Taylor expansion of the polynomial defining the cover. In subsequent work, Matignon [Mat03] generalized the classical Taylor expansion by introducing the notion of p-Taylor expansion, which enabled him to get rid of the cardinality restriction.

Inspired by these ideas, we will now introduce so-called formal p-Taylor expansions; in the context of our new resolution algorithm, these allow to determine all required blow-ups without any of the above limitations. Note that in contrast to the global situation studied by Matignon (that is, covers of the projective line), we are in a purely local setting in which the property of equidistant geometry does not even make sense;

this greater generality also allows us to extend our methods to coverings of prime power degree, see Sect. 5.

For integers $n \in \mathbb{N}_0$, set $v_n := 1 + 1/p + \cdots + 1/p^n \in \mathbb{Q}$. Note that $v_n \to p/(p-1)$ for $n \to \infty$; we will use this fact several times (for example, see Rems. 2.17 and 2.24).

Definition 2.9. Let A be a power series ring over R. A *formal p-Taylor expansion of level $n \in \mathbb{N}_0$* for $f \in A^\times$ is given by a parameter t for A and a function $h \in A^\times$ such that the following condition on the coefficients in the power series expansion

$$f - h^p =: \sum_{i=0}^{\infty} a'_i t^i \in R[\![t]\!]$$

holds true:

$(*)$ $v(a'_{jp}) \geq v_n$ for all $j \in \mathbb{N}_0$.

This is abbreviated to $(f; h, t; a'_i)$.

When considering power series (or in Sect. 4, formal Laurent series) $\sum a'_i t^i$, coefficients a'_i with index of the form $i = pj$, $j \in \mathbb{Z}$, are called *p-coefficients*. In terms of this notion, Def. 2.9 states that a formal p-Taylor expansion $(f; h, t; a'_i)$ approximates f by a pth power in such a way that all p-coefficients of $f - h^p$ are simultaneously made small (with ‘how small’ depending on the level n of the expansion).

Before we can establish the existence of formal p-Taylor expansions, we need a lemma that allows us to uniformly approximate elements of R by pth powers (up to a given level $n \in \mathbb{N}_0$).

Lemma 2.10. *For all $n \in \mathbb{N}_0$, there exists a finite field extension K_n/K such that for all $a \in R$, there is $b \in R_n$ with*

$$v(a - b^p) > v_n + v(a) .$$

(Here R_n denotes the integral closure of R in K_n.)

Proof. For notational convenience, set $K_{-1} := K$ and $R_{-1} := R$. After fixing some uniformizer π_{-1} of R_{-1}, we inductively define for $n \in \mathbb{N}_0$:

$$K_n := K_{n-1}(\pi_n \mid \pi_n^p = \pi_{n-1}) .$$

Then the corresponding rings of integers are $R_n = R_{n-1}[\pi_n \mid \pi_n^p = \pi_{n-1}]$. We show that K_n/K is as in the statement of the lemma; more precisely, for any given $a \in R$, we will inductively construct elements $b^{[n]} \in R_n$ that satisfy

$$v(a - (b^{[n]})^p) > v_n + v(a) . \tag{2.8}$$

For this, we set

$$b^{[-1]} := 0 \in R_{-1} \quad \text{and} \quad b^{[n]} := b^{[n-1]} + \delta^{[n]} \in R_n ,$$

with the improvement term $\delta^{[n]} \in R_n$ defined as follows: as the residue field of R_{n-1} is the algebraically closed field k, we can expand elements of the complete R_{n-1} as power series in the corresponding uniformizer π_{n-1}; in particular, we can write

$$a - (b^{[n-1]})^p =: \sum_{i=0}^{\infty} (a_i^{[n-1]})^p \pi_{n-1}^i ,$$

with $a_i^{[n-1]} \in R_{n-1}$, and then set

$$\delta^{[n]} := \sum_{i=0}^{\infty} a_i^{[n-1]} \pi_n^i \in R_n .$$

We assert that the elements $b^{[n]}$ satisfy inequality (2.8). This will be an easy consequence of the calculation

$$\begin{aligned} a-(b^{[n]})^p &= a-(b^{[n-1]}+\delta^{[n]})^p \\ &= (a-(b^{[n-1]})^p)-(\delta^{[n]})^p-\sum_{j=1}^{p-1}\binom{p}{j}(b^{[n-1]})^j(\delta^{[n]})^{p-j}\,. \end{aligned} \tag{2.9}$$

For the induction basis $n=0$, note that $b^{[-1]}=0$; the first term of (2.9) is therefore just the element a, and the third term vanishes completely. By construction, the second term kills the first term, but only up to mixed terms, which, however, are of valuation strictly larger than

$$1+p\cdot\frac{v(a)}{p}=v_0+v(a)\ :$$

they result as sums of products of a binomial $\binom{p}{j}$ (with $0<j<p$) and p factors of valuation at least $v(a)/p$ (with at least one of these factors having strictly larger valuation).

For the induction step, let $n>0$ be given and assume that $b^{[n-1]}$ fulfills inequality (2.8). Then the first term of (2.9) gives the approximation with respect to level $n-1$. Analog to the case $n=0$, the second term kills the first term up to mixed terms, which are of valuation strictly larger than

$$1+p\cdot\frac{v_{n-1}+v(a)}{p}>1+\frac{v_{n-1}}{p}+v(a)=v_n+v(a)\,,$$

as they result as sums of products of a binomial $\binom{p}{j}$ (with $0<j<p$) and p factors of valuation at least $(v_{n-1}+v(a))/p$ (with at least one of these factors having strictly larger valuation). The summands of the third term have valuation strictly larger than

$$1+j\cdot\frac{v(a)}{p}+(p-j)\cdot\frac{v_{n-1}+v(a)}{p}\geq v_n+v(a)\,.$$

This shows the induction step and finishes the proof. □

Remark 2.11. The assertion of Lem. 2.10 is stronger than needed for the proof of Prop. 2.12; there, it suffices to have an estimation of the form $v(a-b^p)\geq v_n$. Nevertheless, the stronger statement will be needed in Sect. 4.5.1, where we will introduce formal p-Taylor expansions for Laurent series.

The next proposition shows that we can always obtain formal p-Taylor expansions—provided, we allow the base field to get finitely enlarged. In view of Conv. 1.6, this poses no problem; the crucial point is that we can explicitly state a suitable field, making this method also well-suited for our practical, algorithmic purposes.

Proposition 2.12. *Let A be a power series ring over R and let $f\in A^\times$ be given. Fix a level $n\in\mathbb{N}_0$. After replacing K by a suitable finite field extension, there exists, for each parameter t, a formal p-Taylor expansion $(f\,;h,t;a_i')$; moreover, we can assume $a_0'=0$. Both the extension field and the expansion can be determined by a practical algorithm.*

Remark 2.13. With notation as in Lem. 2.10, the proof of Prop. 2.12 will show that an extension field as in the statement of the proposition is given by

$$L:=(\ldots((K(a_0^{1/p})_0)_1)\ldots)_n=K(a_0^{1/p})(\Pi\mid\Pi^{pN}=\pi')\,,$$

with $a_0^{1/p}$ a pth root of the constant coefficient of $f(t)$, with π' a uniformizer of the integral closure of R in $K(a_0^{1/p})$, and with $N = 1 + \cdots + (n+1) = (n+1)(n+2)/2$.

Proof of Prop. 2.12. With respect to the chosen parameter t, we expand $f \in A$ as a power series

$$f = \sum_{i=0}^{\infty} a_i t^i \in R[\![t]\!] \,.$$

Note that $a_0 \in R$ is a unit, as $f \in A$ is assumed to be a unit. By a similar construction as in the proof of Lem. 2.10, we will inductively construct elements $h^{[n]}$, whose pth powers approximate f up to level $n \in \mathbb{N}_0$. In each step, it will be necessary to pass to a successively larger finite extension of K. To avoid confusion with the notation from Lem. 2.10, we denote the respective extensions by $K^{[n]}/K$ and the corresponding rings of integers by $R^{[n]}$; for notational convenience, we also set $K^{[-1]} := K$ and $R^{[-1]} := R$.

For $n \in \mathbb{N}_0$, we define

$$h^{[-1]}(t) := 0 \in R^{[-1]}[\![t]\!] \quad \text{and} \quad h^{[n]}(t) := h^{[n-1]}(t) + \delta^{[n]}(t) \in R^{[n]}[\![t]\!] \,,$$

with the improvement term

$$\delta^{[n]}(t) := \sum_{j=0}^{\infty} b_j^{[n]} t^j \in R^{[n]}[\![t]\!]$$

having coefficients $b_j^{[n]}$, whose pth powers approximate the p-coefficients $a_{jp}^{[n-1]}$ from the previous approximation step

$$f(t) - (h^{[n-1]}(t))^p =: \sum_{i=0}^{\infty} a_i^{[n-1]} t^i \in R^{[n-1]}[\![t]\!]$$

at least up to valuation v_n, that is,

$$v(a_{jp}^{[n-1]} - (b_j^{[n]})^p) \geq v_n \,. \tag{2.10}$$

By Lem. 2.10, all elements $b_j^{[n]}$ as required are contained in the ring of integers $R^{[n]}$ of the finite extension

$$K^{[n]} := (K^{[n-1]})_n$$

of degree p^{n+1} (with notation as in the lemma). Note that other field extensions $K^{[n]}/K^{[n-1]}$ can work as well: it is only necessary that the extensions are finite and contain enough elements for approximations in the sense of (2.10). For instance, when f were a polynomial function (and not a true power series), we could brutally adjoin pth roots

$$b_j^{[n]} := (a_{jp}^{[n-1]})^{1/p}$$

to $K^{[n-1]}$ and use these for the definition of $\delta^{[n]}(t)$: as $f(t)$ would consist of only finitely many terms, the resulting field extension would still be finite (though probably of a larger degree than actually needed); this is noted in Rem. 2.14.

We assert that $h^{[n]}$ gives rise to an approximation of level n; for this, we have to verify that the p-coefficients of $f - (h^{[n]})^p$ have valuation at least v_n. We calculate

$$\begin{aligned} f - (h^{[n]})^p &= f - (h^{[n-1]} + \delta^{[n]})^p \\ &= (f - (h^{[n-1]})^p) - (\delta^{[n]})^p - \sum_{j=1}^{p-1} \binom{p}{j} (h^{[n-1]})^j (\delta^{[n]})^{p-j} \,. \end{aligned} \tag{2.11}$$

For the induction basis $n=0$, note that $h^{[-1]}=0$. Hence, the first term of (2.11) is just f and the third term vanishes completely. By construction, the second term approximates the p-coefficients from the first term with valuation at least v_0, but there are also mixed terms interfering; these, however, are also of valuation at least $v_0=1$: they result as sums of products of a binomial $\binom{p}{j}$ (with $0<j<p$) and p factors of non-negative valuation.

For the induction step, let $n>0$ be given and assume that $h^{[n-1]}\in R^{[n-1]}[\![t]\!]$ satisfies condition $(*)$ of Def. 2.9. Then the first term of (2.11) gives the approximation with respect to level $n-1$. Analog to the case $n=0$, the second term approximates the p-coefficients from the first term with valuation at least v_n, with the additional mixed terms that arise having valuation at least

$$1+p\cdot\frac{v_{n-1}}{p}>v_n\,,$$

as they result as sums of products of a binomial $\binom{p}{j}$ (with $0<j<p$) and p factors of valuation at least v_{n-1}/p. The summands of the third term are seen to be of valuation at least

$$1+(n-j)\cdot\frac{v_{n-1}}{p}\geq v_n\,.$$

As a consequence, $h^{[n]}$ provides an approximation of level n. This shows the induction step.

We still have to see that $h^{[n]}\in R^{[n]}$ is a unit for all $n\in\mathbb{N}_0$. But this is evident because the pth power of the constant coefficient of $h^{[n]}(t)$ approximates the constant coefficient $a_0\in R^\times$ of the unit $f=\sum_{i=0}^\infty a_i t^i\in A^\times$ with strictly positive valuation $v_n>0$ (and therefore is a unit as well). Also, assuming K to contain a pth root $a_0^{1/p}$ of a_0, the approximating function $h^{[0]}(t)=\sum_{j=0}^\infty b_j^{[0]}t^j\in R^{[0]}[\![t]\!]$ of level zero can be defined using this element as constant coefficient, that is, with $b_0^{[0]}:=a_0^{1/p}$. The consequence is that the so-defined approximation kills the constant coefficient of $f(t)$, and we may assume all subsequent approximations of higher level to do so as well (as $b_0^{[0]}$ can be kept as constant coefficient of $h^{[n]}$). In other words, p-Taylor expansions $(f;h,t;a_i')$ can be assumed to satisfy $a_0'=0$. This completes the proof of the proposition. □

Remark 2.14. When f is a polynomial function, the proof of Prop. 2.12 shows that p-Taylor expansions can be defined over finite field extensions that result from K by successively adjoining pth roots of all p-coefficients encountered in the inductive procedure. In particular, this brute force approach is possible for disk coverings coming from the global situation in the course of applying our resolution algorithm (that is, by localizing a p-cyclic covering $Y_R\to X_R$ in critical points as described in Sect. 1.4.3).

Remark 2.15. We will later use approximations like those coming from p-Taylor expansions to determine the minimal exhausting disk. It will then become important to consider the expansions with respect to a suitable parameter, as the minimal exhausting disk is determined not alone by its radius but its center is also of crucial importance. The proof of Lem. 2.35 will show that a good parameter t (that is, a good disk center) has been found when expansions $(f;h,t;a_i')$ with $a_0'=0$ and a_1' small enough exist; the existence of these more special approximations will be established in Prop. 2.31 using a similar construction as above but considering a generic parameter.

We have shown that formal p-Taylor expansions of arbitrary high level exist and can be practically constructed. The next corollary shows that expansions of high enough level give rise to best approximations in the sense of Prop. 2.2.

Corollary 2.16. *Let the power series $f \in A^\times$ be as in the geometric situation (GEOM), and denote by (μ, m) the value of a best possible approximation of f by a pth power in the sense of Prop. 2.2. When the level $n \in \mathbb{N}_0$ of a formal p-Taylor expansion $(f; h, t; a_i')$ is chosen high enough to have $v_n > \mu$ hold true, we have $v_\eta(f - h^p) = (\mu, m)$.*

Remark 2.17. Since $\mu < p/(p-1)$ by Prop. 2.2 and $v_n \to p/(p-1)$ for $n \to \infty$, it is always possible to choose the level $n \in \mathbb{N}_0$ high enough to satisfy the condition of the corollary. In practice, though, we do not know in advance how high the level has to be chosen; yet the method of p-Taylor expansion constitutes a practical algorithm to produce best approximations: finitely many steps will always suffice and by the results of Rem. 2.8, we will be able to recognize a best approximation as such. What is important: this is without any prior assumptions on the base field—the algorithm automatically produces a suitable extension field (in contrast to the algorithm from Prop. 2.2, see Rem. 2.6). This makes Epp's result [Epp73] constructive in the situation under consideration.

Proof of Cor. 2.16. Set $(\tilde{\mu}, \tilde{m}) := v_\eta(f - h^p)$. If $\tilde{m}$ were not prime to p, we would have the chain of inequalities $\tilde{\mu} > v_n > \mu$ due to condition $(*)$ of a p-Taylor expansion and our requirements on the level n. But this is not possible since f can be approximated by a pth power only up to valuation μ, cf. Rem. 2.8. As a consequence, $\tilde{m}$ is prime to p. But then, also by loc.cit., $(\tilde{\mu}, \tilde{m})$ has to coincide with the value (μ, m) of a best possible approximation of f by a pth power. □

Now that we can practically obtain best approximations, we can use Prop. 2.2 to practically describe the ring extension B/A corresponding to the p-cyclic covering $\phi : \mathsf{Y} \to \mathsf{X}$. Further below, when we will use formal p-Taylor expansions to identify the minimal ϕ-exhausting disk, we will need approximations that in some sense are even better, see Cor. 2.23. Namely, we will need approximations that are so good that they also induce a description of the ring extension for the restricted covering $\phi^{-1}(\mathsf{D}) \to \mathsf{D}$, with $\mathsf{D} \subset \mathsf{X}$ an affinoid subdisk. Such approximations will be called sufficiently precise; the exact definition will be given in Sect. 2.2.4.

2.2.3. *Modified Newton Polygon.* The notion of sufficiently precise approximations will be defined in terms of certain Newton polygons. In this section, we recall the basics of this concept.

Definition 2.18. As always, let K be a local field with discrete valuation v. Let a polynomial $f = \sum_{i=0}^n a_i t^i \in K[t]$ be given. The *Newton polygon* of f is defined to be the lower convex hull of the set of points $P_i := (i, v(a_i))$ in the plane, see Fig. 2.1.

The Newton polygon allows to determine the valuation of the zeros of f and their respective multiplicities.

Lemma 2.19. *In the situation of Def. 2.18, let $\mu_j \in \mathbb{Q}$ and $\lambda_j \in \mathbb{N}$ denote the slope resp. the length of the t-axis-projection of the jth line segment in the Newton polygon of f. Then f has precisely λ_j roots of valuation $-\mu_j$ (in the algebraic closure K^{ac} of K).*

Proof. See [Neu02, Satz 6.3]. □

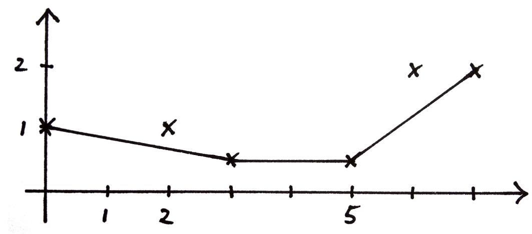

FIGURE 2.1. From the Newton polygon associated to the polynomial $f = 3 + 3t^2 + 3^{1/2}t^3 + 3^{1/2}t^5 + 3^2t^6 + 3^2t^7 \in \mathbb{Q}_3^{\mathrm{ac}}[t]$, we deduce that f has three zeros of valuation $1/6$, two zeros of valuation 0, and two zeros of valuation $-3/4$.

The Newton polygon concept is not limited to polynomials; it can be carried over to the context of formal power series (or formal Laurent series) by defining the Newton polygon as the limit of the Newton polygons for finite partial sums, see [Kob84, Sect. IV.4]. In the following, we need to consider a variant of the Newton polygon, which is well-adapted to our specific situation of p-cyclic coverings. We will make use of the special point $P_0' := (0, p/(p-1))$. Later on, the role of P_0' in determining the correct radius for the minimal exhausting disk will become clear; see Sect. 2.3.1.

Definition 2.20. The *modified Newton polygon* of the power series $f = \sum_{i=0}^{\infty} a_i t^i \in A$ is the lower convex hull over the points $P_i := (i, v(a_i))$, for positive integers $i \in \mathbb{N}$, and the specially defined point

$$P_0 := \begin{cases} (0, v(a_0)) & \text{if } v(a_0) < p/(p-1)\,, \\ P_0' = (0, p/(p-1)) & \text{otherwise}\,. \end{cases}$$

In other words, the modified Newton polygon of f shall start at the point $(0, v(a_0))$ when $v(a_0) < p/(p-1)$ and at $P_0' = (0, p/(p-1))$ otherwise.

For notational convenience, when we refer to a point P_i in the modified Newton polygon, we denote by $v(P_i)$ its value on the $v(\cdot)$-axis. For example, by definition of P_0 in the modified Newton polygon, we always have $v(P_0) \leq p/(p-1)$.

Remark 2.21. It is important to keep in mind that the (modified) Newton polygon of an element $f \in A$ depends on the parameter $t \in A$ chosen, as the (modified) Newton polygon is defined in terms of the power series expansion with respect to that parameter. However, we will skip the reference to the parameter when the risk of misconception is low.

Definition 2.22. If the modified Newton polygon of a power series $f \in A$ has line segments with negative slope, the line segment $\overline{P_l P_k}$ with negative slope of smallest absolute value is called the *critical line segment*; k will be called the *critical index* and P_k the *critical point*.

In the following section, we will define the notion of sufficiently precise approximations in terms of the critical line segment of an approximation.

2.2.4. *Sufficiently Precise Approximation.* The result of Cor. 2.16 can be somewhat strengthened by demanding the level to be even higher.

Corollary 2.23. *Let the power series $f \in A^\times$ be as in the geometric situation (GEOM), and denote by (μ, m) the value of a best possible approximation of f by a pth power in the sense of Prop. 2.2. When the level $n \in \mathbb{N}_0$ of a formal p-Taylor expansion $(f; h, t; a_i')$ with $a_0' = 0$ is chosen high enough to have*

$$\nu_n > \frac{p}{p-1} - \frac{p/(p-1) - \mu}{m}, \tag{2.12}$$

the modified Newton polygon of $(f - h^p)(t)$ contains a critical line segment $\overline{P_l P_k}$ and the following holds true:

(1) *$P_k = P_m = (m, \mu)$,*
(2) *either $p \nmid l$ or $P_l = P_0' = (0, p/(p-1))$.*

Remark 2.24. As $\mu < p/(p-1)$ by Prop. 2.2, the right hand side of inequality (2.12) is also strictly smaller than $p/(p-1)$. Since $\nu_n \to p/(p-1)$ for $n \to \infty$, it is therefore always possible to choose the level $n \in \mathbb{N}_0$ high enough to satisfy the condition of the corollary. Concerning practical applications, the same remarks as in Rem. 2.17 apply: we will recognize when a level high enough has been reached.

Proof of Cor. 2.23. Let the level n of the p-Taylor expansion $(f; h, t; a_i')$ with $a_0' = 0$ be chosen as demanded. We then certainly have $\nu_n > \mu$, so Cor. 2.16 shows that $\nu_\eta(f - h^p) = (\mu, m)$. It follows that the modified Newton polygon of $f - h^p$ has a critical line segment $\overline{P_l P_m}$, with $P_m = (m, \mu)$ being the critical point.

As $a_0' = 0$, the modified Newton polygon of $f - h^p$ starts at the special point $P_0' = (0, p/(p-1))$. Consequently, the absolute value of the slope of the critical segment is at most of value

$$\frac{p/(p-1) - \mu}{m},$$

resulting from connecting $P_0' = (0, p/(p-1))$ and $P_m = (m, \mu)$. If l were a non-zero multiple of p—that is, if we had $l = pj$, with $j \in \mathbb{N}$—we would have $\nu(a_l') > \nu_n$ due to the approximation coming from a p-Taylor expansion of level n; this would result in a line segment too steep—contradiction. Therefore, either $p \nmid l$ or $P_l = P_0'$. □

This motivates the following definition.

Definition 2.25. Let the power series $f \in A^\times$ be as in the geometric situation (GEOM). The approximation of f via $h \in A^\times$ is called *sufficiently precise* with respect to the parameter $t \in A$, if the modified Newton polygon of $(f - h^p)(t)$ contains a critical line segment $\overline{P_l P_k}$ and the following holds true: $p \nmid k$, and either $p \nmid l$ or $P_l = P_0' = (0, p/(p-1))$.

Remark 2.26. By Cor. 2.23, all p-Taylor expansions $(f; h, t; a_i')$ of sufficiently high level and with $a_0' = 0$ give rise to sufficiently precise approximations of f.

Remark 2.27. Sufficiently precise approximations $h \in A^\times$ of $f \in A^\times$ are in particular best approximations: when the critical line segment in the modified Newton polygon of $f - h^p$ is $\overline{P_l P_k}$ then $(\nu(P_k), k)$ is the value of a best approximation of f by a pth power in the sense of Prop. 2.2. As a consequence, the critical point P_k of any sufficiently precise approximation is independent of the parameter $t \in A$ chosen, as we always have $P_k = P_m = (m, \mu)$. To better fit our usual notation, we will therefore most of the time denote the critical point resp. the critical line segment of sufficiently precise approximations by P_m resp. $\overline{P_l P_m}$.

We have the following uniqueness result concerning the critical line segment.

Proposition 2.28. *Let the power series $f \in A^\times$ be as in the geometric situation (GEOM). With respect to a fixed parameter t, all sufficiently precise approximations $h \in A^\times$ of f give rise to the same critical line segment $\overline{P_l P_m}$.*

Proof. Fix a parameter t and denote by $h \in A^\times$ any element giving a sufficiently precise approximation of f by a pth power (with respect to t). Recall that the value (μ, m) of best approximations is uniquely determined and that best approximations can be recognized as such, see Prop. 2.2 and Rem. 2.8. The asserted uniqueness of the critical line segment then follows from applying these facts to the power series $f_\rho := f(p^\rho t)$ and the approximating functions $h_\rho := h(p^\rho t)$, with $\rho \in v(K^{\mathrm{ac}\times})$ and $\rho > 0$.

For this, note that the modified Newton polygon of $f_\rho - h_\rho^p$ is obtained by turning[6] the one of $f - h^p$ counterclockwise, with $P_0' = (0, p/(p-1))$ being the center point. Hence, for small ρ, the critical index of $f_\rho - h_\rho^p$ remains m, and precisely when ρ turns the original critical line segment horizontal—that is, when ρ equals the absolute value of the slope of the original critical line segment—the unique critical index either jumps from m to l (when $l > 0$) or the approximation of f_ρ via h_ρ^p is with value $p/(p-1)$ (when $P_l = P_0'$); see Fig. 2.2 for an illustration of the situation. In any case, the behavior will be independent of the chosen sufficiently precise approximation. □

Remark 2.29. A more technical proof of the above lemma can easily be obtained using a variant of Lem. 4.9 (which is used to prove the analog result of Prop. 2.28 for coverings of open annuli; cf. Prop. 4.7 and its proof).

As a consequence of Prop. 2.28, we can make the following definition.

Definition 2.30. Let the power series $f \in A^\times$ be as in the geometric situation (GEOM). Let $h \in A^\times$ be any sufficiently precise approximation of f with respect to a parameter $t \in A$, and denote the corresponding critical line segment by $\overline{P_l P_m}$. The absolute value of its slope,

$$\rho_0 := |\mathrm{slope}(\overline{P_l P_m})| ,$$

is called the *critical radius* of f with respect to t.

The notion critical 'radius' stems from the fact that the power series f_ρ (notation as in the proof of Prop. 2.28) describes f considered as a function on a subdisk with radius ρ; this will be explained in Sect. 2.3.1. There, we will also see that the critical radius gives the radius of the minimal exhausting disk we are looking for—provided we have chosen a suitable parameter.

Proposition 2.31. *Let $f \in A^\times$ come from the geometric situation (GEOM) and assume the covering given by f via Eq. (2.1) to be non-trivial. After replacing K by a suitable finite field extension, there exists a parameter $t \in A$ such that the critical segment $\overline{P_l P_m}$ of any sufficiently precise approximation of f with respect to t satisfies $l \neq 1$. Both the extension field and the parameter can be determined by a practical algorithm.*

Proof. We fix some parameter t for the power series ring A. Of course, this t will probably not be 'good' in the sense of the proposition to prove. As was briefly mentioned in Rem. 2.15, we intend to use a generic version of our p-Taylor algorithm to deduce

[6] To be precise, it is a shear and not a true rotation.

a suitable parameter. More precisely, the idea is to work with a generic parameter by introducing a new variable T and formally defining

$$F_T(t) := f(t+T) = \sum_{i=0}^{\infty} A_i(T)\, t^i\ , \quad \text{with } A_i(T) \in R[\![T]\!]\ .$$

In the end, we will be able to specialize T appropriately to obtain the desired result. Note that the power series ring $R[\![T]\!]$ is endowed with the Gauß valuation coming from R. In the following, we will have to work with successively larger integral extensions S of $R[\![T]\!]$ (not necessarily finite), which we will endow with an arbitrary but fixed extension of the Gauß valuation. $F_T(t)$ (and all other occuring functions alike) will be considered as power series in t over S.

Analog to the proof of Prop. 2.12, we inductively define approximation power series of $F_T(t)$ by setting

$$H_T^{[-1]}(t) := 0 \quad \text{and} \quad H_T^{[n]}(t) := H_T^{[n-1]}(t) + \Delta_T^{[n]}(t)\,, \tag{2.13}$$

with the improvement term

$$\Delta_T^{[n]}(t) := \sum_{j=0}^{\infty} B_j^{[n]}(T)\, t^j \tag{2.14}$$

having coefficients

$$B_j^{[n]}(T) := (A_{jp}^{[n-1]}(T))^{1/p}\ , \tag{2.15}$$

which are pth roots of the p-coefficients from the previous approximation step

$$F_T(t) - (H_T^{[n-1]}(t))^p =: \sum_{i=0}^{\infty} A_i^{[n-1]}(T)\, t^i\ . \tag{2.16}$$

Note that each step usually requires to pass to successively bigger integral extensions S of $R[\![T]\!]$. Due to the brute force like approach, those extensions might well be of infinite degree; when f is a polynomial function, however, finite extensions suffice. Also note that $H_T^{[n]}(t)^p$ kills the constant coefficient of $F_T(t)$ because $H_T^{[n]}(t)$ has by construction a pth root of $A_0(T)$ as constant coefficient. With $A_0(T)$ being a unit of S, it follows that $H_T^{[n]}(t)$ is a unit of $S[\![t]\!]$.

Exactly as in the non-generic calculations, we obtain from the equations

$$\begin{aligned} F_T - (H_T^{[n]})^p &= F_T - (H_T^{[n-1]} + \Delta_T^{[n]})^p \\ &= (F_T - (H_T^{[n-1]})^p) - (\Delta_T^{[n]})^p - \sum_{j=1}^{p-1} \binom{p}{j} (H_T^{[n-1]})^j (\Delta_T^{[n]})^{p-j} \end{aligned}$$

that $H_T^{[n]}$ gives rise to an approximation for F_T of level n (with respect to the chosen valuation on S); see the proof of Prop. 2.12. Specializing T will yield an approximation for the initial function f. Such a specialization map $S \to R^{\mathrm{ac}}$ can be defined as follows: For arbitrary $\xi \in \mathfrak{m}^{\mathrm{ac}}$, we get an R-homomorphism $R[\![T]\!] \to R^{\mathrm{ac}}$ by sending T to ξ; note that we need ξ to lie in the maximal ideal $\mathfrak{m}^{\mathrm{ac}} \lhd R^{\mathrm{ac}}$ for the image series to converge. Then $S \otimes_{R[\![t]\!]} R^{\mathrm{ac}}$ is an integral extension of R^{ac} and therefore coincides with the integrally closed R^{ac}. It follows that the homomorphism $R[\![T]\!] \to R^{\mathrm{ac}}$ can be extended to an R-homomorphism $S \to R^{\mathrm{ac}}$. The resulting function $h := H_\xi^{[n]}(t)$ will provide an approximation for f of level n and with respect to the new parameter $t' := t - \xi$. As $h \in R^{\mathrm{ac}}[\![t']\!]$ will usually not be defined over R—unless, for example, f is polynomial—this is not a p-Taylor expansion in the original sense of the definition.

Nevertheless, the approximation property (∗) of Def. 2.9 is satisfied, and we will use the usual notation $(f;h,t';a_i')$. Note that $a_0' = 0$ by construction. The proof of Cor. 2.23 shows that, when the level $n \in \mathbb{N}_0$ is chosen high enough, this approximation will be sufficiently precise in the sense that the critical line segment in the modified Newton polygon of $(f - h^p)(t')$ satisfies the conditions from Def. 2.25. Note that the required level can be estimated as in the statement of Cor. 2.23—and that this can be done prior to the current process of finding a good parameter.

We would like to find $\xi \in \mathfrak{m}^{\mathrm{ac}}$ leading to a small enough first coefficient $A_1^{[n]}(\xi)$. To this end, observe that by the inductive definition (2.13)–(2.16) of the approximating functions, the coefficients $A_i^{[n]}(T)$ depend on only the first ip terms of the previous approximation step $F_T - (H_T^{[n-1]})^p$. As a consequence, for finite level $n \in \mathbb{N}_0$, the coefficient $A_1^{[n]}(T)$ can be defined over a finite extension $\tilde{S}/R[\![T]\!]$. We can then consider its norm

$$a(T) := \mathrm{Norm}_{\tilde{S}/R[\![T]\!]}(A_1^{[n]}(T)) \in R[\![T]\!] \ .$$

By Weierstraß preparation, we have $a = bPu$, with an element $b \in R$, a distinguished polynomial $P \in R[T]$, and an invertible power series $u \in R[\![T]\!]^\times$. We would like to see that P is non-constant, as we could then define a specialization map sending T to a zero $\xi \in R^{\mathrm{ac}}$ of the distinguished polynomial P, with the consequence that $A_1^{[n]}(T)$ would be send to zero as well. The resulting sufficiently precise p-Taylor-like expansion $(f;h,t';a_i')$ would then satisfy $a_1' = 0$, so that the corresponding critical line segment could not involve the point P_1. Because best approximations remain so under base change to arbitrary algebraic extensions (Rem. 2.8), the uniqueness result from Prop. 2.28 carries over to approximations defined over R^{ac}. As a consequence, the critical segment $\overline{P_l P_m}$ of any true sufficiently precise approximation of f with respect to the so-determined parameter t' would then also satisfy $l \neq 1$. In practice, such approximations could be obtained (after finitely enlarging the base field K) using, for example, the Taylor algorithm from Prop. 2.12. Consequently, when the Weierstraß polynomial P happens to be non-constant (which is the case for the calculations in Sect. 3), we are finished with the proof.

Unfortunately, without additional assumptions on f, the polynomial P is not always non-constant. It is, however, when we are in the situation where $h = 1$ provides a best possible approximation (μ, m) of f: Recall that in the case of a power series ring, the second component $\#_\eta$ of the rank-two-valuation v_η gives the Weierstraß order. The crucial fact that P is non-constant would therefore follow if we could show

$$\#_\eta(A_1^{[n]}(T)) \geq 1, \tag{2.17}$$

as this would imply $\#_\eta(a(T)) \geq 1$.[7] Under the above assumption that $h = 1$ gives a best approximation, we can see by an inductive argument that inequality (2.17) holds true. For this, note that $A_1^{[-1]}(T) = f'(T)$ is the derivative of $f(T)$ and that, in the current situation, $v_\eta(f') = (\mu, m-1)$. Since f describes a non-trivial covering, we have $m > 1$ by Cor. 2.5, and therefore $\#_\eta(A_1^{[-1]}(T)) \geq 1$ as desired. Subsequent approximation steps do not change this behavior and we retain $\#_\eta(A_1^{[n]}(T)) \geq 1$, for all $n \in \mathbb{N}_0$. This is because $A_1^{[n]}(T)$ results from $A_1^{[n-1]}(T)$ by adding terms of valuation strictly larger

[7]Admittedly, we slightly abuse notation here by also writing $\#_\eta$ for some extension to $\tilde{S}$.

than μ: indeed, we deduce from the approximation formulas (2.13)–(2.16) that the new first coefficient is

$$A_1^{[n]}(T) = A_1^{[n-1]}(T) - p\,B_0^{[0]}(T)^{p-1}\,B_1^{[n]}(T)\,,$$

where $B_0^{[0]}(T) = A_0(T)^{1/p}$ is a unit, and $B_1^{[n]}(T)$ has valuation at least μ/p; since $\mu < p/(p-1)$ holds true, we have $1+\mu/p > \mu$. The terms subtracted from $A_1^{[n-1]}(T)$ therefore have no influence on its Weierstraß order—it remains $m-1 \geq 1$. We deduce that P is non-constant in the situation where $h = 1$ provides a best approximation.

It remains to show how to proceed when P is constant. In this case, take any best approximation $h_1 \in A^\times$ of f and set $f_2 := f/h_1^p$. In practice, this pre-approximation of f can be done (after a finite extension of the base field) by means of a sufficiently precise formal p-Taylor expansion with respect to the parameter t we started with, see Rem. 2.27. Since 1 is a best possible approximation of f_2, the reasoning from above shows that (after a further finite extension of the base field) there is a parameter $t' \in A$ and a sufficiently precise approximation $h_2 \in A^\times$ of $f_2(t')$ with a critical line segment $\overline{P_l P_m}$ satisfying $l \neq 1$. But then $h := h_1 h_2$ provides the desired approximation of the initial function f (with respect to t'): the line segments with negative slope in the modified Newton polygons of $f - h^p$ and $f_2 - h_2^p$ coincide, as the former power series results from the latter by multiplication with the unit $h_1^p \in A^\times$. In particular, the approximation of $f(t')$ via h is sufficiently precise and has a critical line segment $\overline{P_l P_m}$ with $l \neq 1$. □

Remark 2.32. The proof of Prop. 2.31 shows that, when f is polynomial (which happens, for example, when the disk covering is induced by a global p-cyclic covering through localizing in critical points), the generic p-Taylor algorithm can be used to produce sufficiently precise approximations of f over a finite extension of R. If in this case $\#_\eta(A_1^{[n]}(T))$ happens to be strictly positive, both a suitable parameter and a sufficently precise approximation as in the statement of the proposition can be determined at once (without the need to do an additional pre-approximation step). This is the situation in the example treated in Sect. 3.

2.3. **Minimal Exhausting Disk.** We can now turn to our main task of determining the minimal exhausting disk for p-cyclic disk coverings.

Proposition 2.33. *Let the situation be as in Sect. 2.1; that is, $f \in A^\times$ shall be as in (GEOM) and define a non-trivial p-cyclic covering $\phi : Y \to X$ of the open unit disk. After replacing K by a suitable finite field extension, Prop. 2.31 guarantees the existence both of a parameter $t \in A$ and of a sufficiently precise approximation $h \in A^\times$ of $f(t)$ with a critical line segment $\overline{P_l P_m}$ satisfying $l \neq 1$. Denote by $\rho_0 = |\mathrm{slope}(\overline{P_l P_m})|$ the corresponding critical radius. Then the affinoid disk*

$$D := \{x \in X \mid \rho_0 \leq v(t(x))\}$$

is the minimal exhausting disk for the covering ϕ.

The proof will occupy the rest of Sect. 2.3.

2.3.1. *Exceptional Divisor.* A first step towards proving Prop. 2.33 is to understand what effect the modification induced by D has on the formal model of Y. As explained in Sect. 1.4.4, the exceptional divisor corresponds to the reduction of the affinoid

$$\mathsf{E} := \phi^{-1}(\mathsf{D})\,.$$

We therefore have to study the rigid analytic map

$$\phi_1 := \phi|_{\mathsf{E}} : \mathsf{E} \to \mathsf{D}$$

and the corresponding cover of formal models $\mathrm{Spf} B_1 \to \mathrm{Spf} A_1$, where $B_1 := \mathring{\mathscr{O}}_{\mathsf{E}}$ and $A_1 := \mathring{\mathscr{O}}_{\mathsf{D}}$ denote the respective rings of zero-bounded analytic functions.

Replacing K by a finite extension (if necessary), we obtain a parameter t_1 for D by appropriately rescaling the disk parameter of X by setting $t_1 := p^{-\rho_0} t$, where p^{ρ_0} shall denote some element of K with valuation ρ_0. In terms of t_1, we can describe A_1 as the Tate algebra $A_1 = R\langle t_1 \rangle$. The restricted covering ϕ_1 is still given by the Kummer equation (2.1), but now considered as an equation over A_1. As before, we would obtain a description of the ring extension B_1/A_1 if we could approximate f by a pth power such that a variable change of the form (2.4) would lead to an equation remaining irreducible in reduction. Note that this approximation has to be done using the Gauß valuation v_{ρ_0} on D (that is, the valuation on A_1 with respect to t_1). We will see below that the desired approximation is accomplished by the sufficiently precise approximation $h \in A^\times$ of f; we write $f - h^p =: \sum_{i=0}^\infty a_i' t^i$.

The power series expansion of $f - h^p$ with respect to $t_1 = p^{-\rho_0} t$ is given by

$$f - h^p = \sum_{i=0}^{\infty} a_i' p^{i\rho_0} t_1^i \; ;$$

note that this coincides with the power series denoted $f_{\rho_0} - h_{\rho_0}^p$ in the proof of Prop. 2.28. It follows that

$$v_{\rho_0}(f - h^p) = \mu + m\rho_0$$

holds true. More precisely, a coefficient $a_i' p^{i\rho_0}$ of the expansion with respect to t_1 has minimal valuation $\mu + m\rho_0$ precisely when the point $P_i = (i, v(a_i'))$ lies on the critical line segment $\overline{P_l P_m}$ of the approximation $(f - h^p)(t)$ (all other terms have strictly larger valuation); cf. Fig. 2.2. Also, by definition of the critical radius, we have

$$\mu + m\rho_0 \begin{cases} < p/(p-1) & \text{if } l > 0 \,, \\ = p/(p-1) & \text{if } P_l = P_0' \,. \end{cases}$$

In a way, this is the reason for defining the modified Newton polygon using the special point $P_0' = (0, p/(p-1))$, as the critical radius will then lead to approximations with valuation at most $p/(p-1)$. The latter is the maximal value that makes sense to consider, as better approximations would imply that $\mathsf{E} = \phi^{-1}(\mathsf{D})$ would decompose into p copies of D (as $f(t_1) \in A_1$ were then a pth power). Since there is only one boundary above the boundary of X (Cor. 2.4), D could then not be exhausting.

We assume K to contain some element λ_1 of valuation $v(\lambda_1) = (\mu + m\rho_0)/p$. We can then write

$$f - h^p = \lambda_1^p \sum_{i=0}^{\infty} b_i t_1^i \,,$$

with coefficients $b_i = a_i' p^{i\rho_0} / \lambda_1^p \in R$. The variable change

$$w_1 := \frac{y - h}{\lambda_1}$$

then leads to the equation

$$w_1^p + \cdots + p\lambda_1^{1-p} h^{p-1} w_1 = \frac{f - h^p}{\lambda_1^p} \tag{2.18}$$

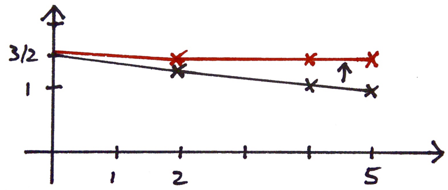

FIGURE 2.2. When restricting a covering of the open unit disk to the preimage of a closed subdisk with radius $\rho_0 > 0$, the equation defining the original covering has to be rewritten in terms of the scaled parameter $t_1 = p^{-\rho_0}t$; this has the effect of shearing the corresponding modified Newton polygon. The drawing is for Eq. (3.6) from the example treated in Sect. 3.

over A_1. Since all coefficients b_i with index $i < l$ or index $i > m$ have strictly positive valuation, Eq. (2.18) reduces modulo (π) to the polynomial equation

$$\bar{w}_1^p + \bar{c}\bar{w}_1 = \bar{b}_l\bar{t}_1^l + \cdots + \bar{b}_m\bar{t}_1^m \tag{2.19}$$

over $k[\bar{t}_1]$. Here we have denoted the reduction of $p\lambda_1^{1-p}h^{p-1}$ by $\bar{c}$. This is a non-zero element precisely when $v(\lambda_1) = 1/(p-1)$ (that is, when $P_l = P_0'$). It follows that the reduction is given either by an Artin–Schreier type equation (when $P_l = P_0'$) or by a purely inseparable equation (when $l > 0$). In any case, since m is prime to p (as the approximation via h is sufficiently precise), Eq. (2.19) is irreducible and hence so is also Eq. (2.18). By the same arguments as in the proof of Prop. 2.2 (using Serre's criterion on normality), we deduce that w_1 generates B_1 over A_1. As a consequence, the exceptional divisor of the modification induced by D—corresponding to the special fiber of $\operatorname{Spf} B_1$—is described by the reduced equation (2.19).

2.3.2. *Proof of the Proposition.* We are now in a position to prove Prop. 2.33. The proposition is a direct consequence of the next two lemmata, which we will prove using the preliminary considerations from Sect. 2.3.1 above. □

Lemma 2.34. *With notation as in Prop. 2.33, $D \subset X$ is ϕ-exhausting.*

Proof. To show that D is exhausting—or equivalently, to show that $\mathsf{A} := \mathsf{X} \setminus \mathsf{D}$ is a separating boundary domain—we have to see that $\mathsf{V} := \phi^{-1}(\mathsf{A}) \subset \mathsf{Y}$ is a disjoint union of open annuli. By Cor. 2.4, there is only one boundary point $\xi \in \partial\mathsf{Y}$ lying above $\eta \in \partial\mathsf{X}$ (the unique boundary point of the open disk). Hence, if A were to be separating, V would necessarily be a single open annulus.

Denote the rings of zero-bounded analytic functions on V and A by $B_2 := \mathring{\mathcal{O}}_{\mathsf{V}}$ and $A_2 := \mathring{\mathcal{O}}_{\mathsf{A}}$, respectively. The restricted covering $\psi := \phi|_{\mathsf{V}} : \mathsf{V} \to \mathsf{A}$ corresponds to the ring extension B_2/A_2, where B_2 is determined as the integral closure of A_2 in the extension of $\operatorname{Frac} A_2$ given by the Kummer equation (2.1) (now considered as an equation over A_2). As

$$\mathsf{A} = \{x \in \mathsf{X} \mid v(t(x)) < \rho_0\},$$

we have $A_2 = R[\![s,t \mid st = p^{\rho_0}]\!]$. We want to apply Lem. 1.31 with respect to the covering ψ and deduce that V is an open annulus.

For this, we first have to check that there is also only one boundary point $\xi_2 \in \partial\mathsf{V}$ lying above $\eta_2 \in \partial\mathsf{A}$ (the boundary point of A distinct from the one coming from the disk X). The boundary η_2 corresponds to the height-one prime ideal $\mathfrak{p}_2 := (\pi, t) \lhd A_2$. Denoting the corresponding discrete valuation on $\operatorname{Frac} A_2$ by $v_{\mathfrak{p}_2}$, we need to see that there is only one extension of $v_{\mathfrak{p}_2}$ to $\operatorname{Frac} B_2$. But this is easy by what we have already seen: Writing Eq. (2.1) in terms of $s = p^{\rho_0} t^{-1}$ (which serves as a parameter of A with respect to η_2), we are led to an equation analog to Eq. (2.18); its reduction

$$\bar{w}_1^p + \bar{c}\bar{w}_1 = \bar{b}_m \bar{s}^{-m} + \cdots + \bar{b}_l \bar{s}^{-l}$$

describes an extension of the residue field $\operatorname{Frac} k[\![\bar{s}]\!]$ of $v_{\mathfrak{p}_2}$ with degree $f = p$. We deduce from the fundamental equality $p = efg$ that $v_{\mathfrak{p}_2}$ can have only one extension to $\operatorname{Frac} B_2$ and, accordingly, that only one boundary point $\xi_2 \in \partial\mathsf{V}$ lies above $\eta_2 \in \partial\mathsf{A}$.

In a next step, we have to find an element $w \in B_2$ satisfying conditions (1) and (2) of Lem. 1.31. Let $h \in A^\times$ be the sufficiently precise approximation of f from the statement of Prop. 2.33, and denote the corresponding critical line segment by $\overline{P_l P_m}$, with $P_m = (m, \mu)$. Assume K to be large enough to contain $\lambda \in K$ with $v(\lambda) = \mu/p$. We assert that the element

$$w := \frac{y - h}{\lambda} \in B \subset B_2 \tag{2.20}$$

has the required properties.

Observe that the Kummer equation (2.1) remains irreducible over the larger ring $A_2 \supset A$. Indeed, we have seen above that the extension of valuations v_{ξ_2}/v_{η_2} has residue degree $f = p$. As the valuation ring of v_{ξ_2} dominates B_2, the field extension $\operatorname{Frac} B_2 / \operatorname{Frac} A_2$ also needs to have degree p, and therefore Eq. (2.1) has to be irreducible over A_2. As a consequence, the equation

$$w^p + \cdots + p h^{p-1} \lambda^{1-p} w = \frac{f - h^p}{\lambda^p},$$

resulting from the variable change (2.20), is irreducible as well and therefore is the minimal polynomial of w. We can hence calculate

$$\operatorname{Norm}(w) = \frac{f - h^p}{\lambda^p} =: g .$$

By Weierstraß preparation, we obtain a decomposition $g = Pu$, with a distinguished polynomial $P = c_1 t + \cdots + t^m \in R[t]$ of degree m and an invertible power series $u \in R[\![t]\!]$. We express P in terms of t and $s = p^{\rho_0} t^{-1}$, writing

$$P = t^m (c_1 t^{1-m} + \cdots + 1) = t^m (c_1 p^{\rho_0(1-m)} s^{m-1} + \cdots + 1) .$$

It follows from $g = Pu$, with u a unit, that the line segments with negative slope in the modified Newton polygons of $P(t)$ and $g(t)$ coincide. Note that the critical radius ρ_0 is smaller or equal to the absolute value of the slope of any of these segments because, by definition, ρ_0 corresponds to the line segment with negative slope of smallest absolute value in the modified Newton polygon of $f - h^p = \lambda^p g$. As a consequence, all coefficients $c_i p^{\rho_0(i-m)}$, with $0 < i < m$, have non-negative valuation and are therefore elements of R. But then $1 + \cdots + c_1 p^{\rho_0(1-m)} s^{m-1}$ is a unit of $R[\![s]\!]$ and, a fortiori, also a unit of $A_2 = R[\![s, t \mid st = p^{\rho_0}]\!]$. It follows that w satisfies condition (1) of Lem. 1.31. As the critical index m is prime to p, condition (2) is also satisfied.

We can thus apply the lemma and therewith conclude that A is separating, or in other words, that D is exhausting. □

Lemma 2.35. *With notation as in Prop. 2.33, $\mathsf{D} \subset \mathsf{X}$ gives rise to an improving modification.*

Proof. As set out in Sect. 2.3.1, the exceptional divisor of the modification induced by the closed disk D is described by Eq. (2.19), which is either an Artin–Schreier equation (in the case $P_l = P_0'$) or a purely inseparable equation (in the case $l > 0$).

In the former case, the exceptional divisor is a smooth component of positive genus (note that $m > 1$ by Cor. 2.5). In the latter case, we have to use our assumption on the chosen parameter t; namely, t is such that the sufficiently precise approximation h gives rise to a critical line segment $\overline{P_l P_m}$ with $P_l \neq P_1$. We therefore even have $l > 1$. It follows that the differential of the right hand side of the purely inseparable Eq. (2.19) has at least two distinct zeros; as a consequence of the Jacobian criterion of regularity, the exceptional divisor then has at least two distinct singularities not on the strict transform. In any case, Lem. 1.32 shows that D corresponds to an improving modification. □

Having constructed the minimal exhausting disk for the covering $\phi : \mathsf{Y} \to \mathsf{X}$ from Sect. 2.1, we have also proven Thm. 2.1. □

Remark 2.36. We wish to emphasize once more the suitability of the above approach for practical purposes. Determining a sufficiently precise approximation means to determine a good radius for the minimal exhausting disk; determining a parameter in the sense of Prop. 2.31 means to determine a center point of the minimal exhausting disk. As both can be done explicitly with the method of formal p-Taylor expansion using both the non-generic and the generic version of the algorithm (Props. 2.12 and 2.31)—and without any prior assumptions on the base field K (Rems. 2.24 and 2.26)—we are able to explictly construct the minimal exhausting disk.

3. Example: Equidistant Prime-Cyclic Galois Cover

Let us illustrate the methods from Sect. 2 by calculating in detail the stable reduction of the covering

$$\Phi : Y \to X := \mathbb{P}^1_{\mathbb{Q}_3}$$

that was mentioned at the beginning of Sect. 2.2: the covering of the projective line over the 3-adic field $\mathbb{Q}_3$ shall be given by the Kummer type equation

$$y^3 = 1 + 3t^3 + 3t^5 =: f(t)\,. \tag{3.1}$$

It follows from the Riemann–Hurwitz formula [Har77, Cor. IV.2.4] that Y is a curve of genus $g_Y = 4$.

Note that the covering has the equidistant geometry property: the five zeros of f have valuation $-1/5$ and their pairwise distance is of valuation 1, as can be deduced from the Newton polygons of $f(t)$ and $f(t+a)/t$, where $a \in \mathbb{Q}_3^{\mathrm{ac}}$ denotes any zero of $f(t)$. But this is irrelevant for the computations to follow since our algorithm does not rely on this fact. It will become apparent, however, that even in this basic example the calculations involved are highly non-trivial.

For realizing the semistable reduction of Y, we will have to pass to a sufficiently large finite field extension of $\mathbb{Q}_3$. To make notation in the upcoming calculations more transparent, we will from now on consider Y as a K-curve, where $K/\mathbb{Q}_3$ is a fixed but large enough finite field extension containing all elements that we need in the course of applying our algorithm; the ring of integers of K is denoted by R, the (finite) residue field by k. Keeping track of the field elements that we actually need, we will a posteriori be able to determine the minimal field extension over which a semistable model of Y can be realized.

3.1. **From Global to Local.** We start our study by taking the standard smooth model $X_R := \mathbb{P}^1_R$ for X. We need to get an equation describing the corresponding model Y_R (defined as the normalization of X_R in $K(Y)$) to be able to determine the critical points of the covering, that is, the points where our resolution algorithm has to be applied.

Equation (3.1) reduces modulo 3 to the third power $\bar{y}^3 = 1$, which describes a non-reduced scheme. Therefore, y does not generate the ring extension corresponding to the affine part $\operatorname{Spec} R[t] \subset X_R$ and its Φ-preimage, since we know Y_R to be permanent due to our assumption that K is large enough. We thus have to find a better suited element. Assuming K is large enough to contain an element $\lambda \in K$ with valuation $v(\lambda) = 1/3$, we substitute $w := (y-1)/\lambda$ and obtain

$$w^3 + 3\lambda^{-1}w^2 + 3\lambda^{-2}w = 3\lambda^{-3}(t^3 + t^5)\,.$$

This equation is well-suited, as it has reduced and irreducible reduction of the form

$$\bar{w}^3 = \bar{t}^3 + \bar{t}^5 \tag{3.2}$$

and therefore describes the ring extension corresponding to the considered part of the model; cf. the proof of Prop. 2.2 and the reasoning using Serre's criterion. The differential of the right hand side of the purely inseparable Eq. (3.2) is $\mathrm{d}(\bar{t}^3 + \bar{t}^5) = 5\bar{t}^4$; its only zero—corresponding to a single critical point—is at $\bar{t} = \bar{0}$. As $\bar{t} = \infty$ is seen to be not critical, there is exactly one point where our resolution algorithm has to be applied. Namely, we need to study the residue class

$$\mathsf{X} := \,]\bar{0}[_{X_R}\,,$$

which is a rigid analytic open unit disk, and the induced covering thereof—that is, the cover of open analytic curves

$$\phi : \mathsf{Y} := \Phi^{\mathrm{rig}^{-1}}(\mathsf{X}) \to \mathsf{X}$$

given by the same equation (3.1) as before, but now considered as an equation over the completed ring $R[\![t]\!] = \mathring{\mathcal{O}}_{\mathsf{X}}$. Note that t serves as a disk parameter for X.

3.2. **Applying the Algorithm.** Because ϕ is a non-trivial 3-cyclic étale Galois cover of the rigid analytic open unit disk, we are in the situation of Sect. 2; consequently, we can apply our techniques from there to determine all relevant modifications.[8]

3.2.1. *Suitable Center.* For a first improvement of the model, we have to find the minimal exhausting disk with respect to ϕ. To begin with, we calculate the generic 3-Taylor expansion of f by following the lines of the proof of Prop. 2.31. As f is polynomial (and not a true power series), this approach—though brute force like—will involve only finite extensions, cf. Rem. 2.32.

We hence introduce a new variable T and examine

$$\begin{aligned} F_T(t) &:= f(t+T) \\ &= (1+3T^3+3T^5) + (9T^2+15T^4)\,t + (9T+30T^3)\,t^2 \\ &\quad + (3+30T^2)\,t^3 + 15T\,t^4 + 3t^5\,, \end{aligned} \tag{3.3}$$

[8]In order that ϕ is a Galois cover, the field K needs to contain a primitive third root of unity (which we could assume); however, for our techniques from Sect. 2 to be applicable, it is only important that the covering is given by an equation of Kummer type (as is the case here).

which we interpret as a power series in t over the power series ring $R[\![T]\!]$. Passing to the integral degree-nine-extension $S/R[\![T]\!]$ by adjoining elements α_T and β_T with

$$\alpha_T^3 = 1 + 3T^3 + 3T^5 \quad \text{and} \quad \beta_T^3 = 3 + 30T^2 ,$$

we can set

$$H_T(t) := \alpha_T + \beta_T t ,$$

therewith obtaining the approximation

$$F_T - H_T^3 = (9T^2 + 15T^4 - 3\alpha_T^2\beta_T)\,t + (9T + 30T^3 - 3\alpha_T\beta_T^2)\,t^2 + 15T\,t^4 + 3t^5 ,$$

which is sufficiently precise as no third powers of t occur at all. The norm of the first coefficient (with respect to $S/R[\![T]\!]$) is the third power of the polynomial

$$m(T) := 3+30T^2+18T^3+198T^5+180T^7+189T^8+342T^{10}+145T^{12} , \quad (3.4)$$

which is of Eisenstein type over $\mathbb{Q}_3$. In particular, this is a non-constant distinguished polynomial over R, so we are lucky in that we do not need a pre-approximation step to be able to determine a suitable disk center; see the proof of Prop. 2.31 and Rem. 2.32.

We assume K to be large enough to contain a zero $\xi \in R$ of $m(T)$; this zero will be a suitable center point for the minimal exhausting disk. Keeping track of the elements required in the course of our algorithm, we note that K has to contain

$$K_1 := \mathbb{Q}_3(\xi \mid m(\xi) = 0) ,$$

which is a totally ramified extension field of degree twelve (and as such also contains an element λ of valuation $1/3$, as was needed in the very first step in Sect. 3.1).

3.2.2. *First Improvement.* We specialize Eq. (3.3) accordingly, sending T to ξ, to obtain an equation for the covering ϕ with respect to the better suited parameter $t_1 := t - \xi$:

$$\begin{aligned} y^3 = F_\xi(t_1) = (1 + 3\xi^3 + 3\xi^5) + (9\xi^2 + 15\xi^4)\,t_1 + (9\xi + 30\xi^3)\,t_1^2 \\ + (3 + 30\xi^2)\,t_1^3 + 15\xi t_1^4 + 3t_1^5 . \end{aligned} \quad (3.5)$$

Then

$$h := H_\xi(t_1) = \alpha_\xi + \beta_\xi t_1 ,$$

with third roots α_ξ of $1 + 3\xi^3 + 3\xi^5$ and β_ξ of $3 + 30\xi^2$, gives a sufficiently precise 3-Taylor expansion of f with respect to the parameter t_1. This is provided β_ξ is chosen in accordance with ξ and α_ξ: the requirement, that the first coefficient of $f - h^3$ (considered as a power series in t_1) shall be killed, determines which of the three possible roots for β_ξ has to be taken, in that

$$\beta_\xi = \frac{9\xi^2 + 15\xi^4}{3\alpha_\xi^2}$$

has to hold true. As a consequence, α_ξ generates the field

$$K_2 := K_1(\alpha_\xi, \beta_\xi) = K_1(\alpha_\xi \mid \alpha_\xi^3 = 1 + 3\xi^3 + 3\xi^5)$$

over K_1; the extension is of degree three and totally ramified. The field K, not specified in detail, has to contain K_2 as a subfield. We obtain the expansion

$$\begin{aligned} f - h^3 = (&4860 + 62496\,\xi + 40500\,\xi^2 + 402468\,\xi^3 - 135000\,\xi^4 \\ &+ 223560\,\xi^5 + 256833\,\xi^6 - 81000\,\xi^7 + 604299\,\xi^8 \\ &- 573750\,\xi^9 + 140940\,\xi^{10} - 326250\,\xi^{11})/4097\; t_1^2 \\ &+ 15\,\xi t_1^4 + 3t_1^5\,. \end{aligned} \tag{3.6}$$

By construction, there is no linear term.

The coefficients of $f - h^3$ (with respect to t_1) have valuation[9]

$$\left[\infty, \infty, \frac{45}{36}, \infty, \frac{39}{36}, \frac{36}{36}\right]\;;$$

the critical index and its valuation is thus $(\mu, m) = (1, 5)$, and the critical line segment is given by $\overline{P_2P_5} = \overline{(2, 45/36)(5, 36/36)}$ (depicted in Fig. 2.2). We calculate the critical radius to be

$$\rho_0 := \frac{45/36 - 36/36}{5 - 2} = \frac{1}{12}\,.$$

Proposition 2.33 then implies that

$$\mathsf{D} := \{x \in \mathsf{X} \mid \rho_0 \le v(t_1(x))\}$$

is the minimal ϕ-exhausting disk. As explained in Sects. 1.4.4 and 2.3.1, the modification induced by D is described by restricting ϕ to this smaller closed disk. On the level of equations, this means to express Eq. (3.5) in terms of

$$t_2 := \frac{t_1}{\xi} = \frac{t - \xi}{\xi}\,,$$

which serves as a parameter for D. Note that $v(\xi) = \rho_0$ and that we could take the initial center (defined by $t = 0$) as a center for D as well; for the calculations to follow, however, t_2 is better suited. For the restricted covering $\phi^{-1}(\mathsf{D}) \to \mathsf{D}$, we obtain the equation

$$\begin{aligned} y^3 = (1 + 3\xi^3 + 3\xi^5) + (9\xi^3 + 15\xi^5)\,t_2 + (9\xi^3 + 30\xi^5)\,t_2^2 \\ + (3\xi^3 + 30\xi^5)\,t_2^3 + 15\xi^5 t_2^4 + 3\xi^5 t_2^5 \end{aligned} \tag{3.7}$$

over $R\langle t_2\rangle$.

With respect to the rank-two-valuation on X, the approximation of f by h^3 is of value $(\mu, m) = (1, 5)$; the approximation with respect to the Gauß valuation v_{ρ_0} on D is therefore of value $1 + 5 \cdot 1/12 = 17/12$. As the element $3\xi^5 \in K_2$ has valuation $v(3\xi^5) = 17/36$, the usual variable change

$$w_1 := \frac{y - h}{3\xi^5}$$

leads to an equation suitable for reduction: we obtain a description of the exceptional divisor in terms of the purely inseparable extension of $k[\bar{t}_2]$ given by

$$\bar{w}_1^3 = \bar{t}_2^2 + 2\bar{t}_2^4 + \bar{t}_2^5\,. \tag{3.8}$$

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[9] As always, the valuation is normalized such that the valuation of the prime characteristic p is $v(p) = 1$; in the current case, $v(3) = 1$.

The terms not killed in the reduction process correspond to points on the original critical line segment (which is turned horizontal when considering the equation over the smaller disk D); Figure 2.2 shows the modified Newton polygons.

The differential $2(\bar{t}_2+\bar{t}_2^3+\bar{t}_2^4)\,d\bar{t}_2$ of the right hand side of Eq. (3.8) has four distinct zeros in k, as we can assume K to be large enough to have k containing $\mathbb{F}_9$: this means that K has to contain

$$K_3 := K_2(u \mid u^2+2u+2=0)\,, \tag{3.9}$$

the unique unramified extension of K_2 of degree two. There are then four critical points, and in each of these, our algorithm needs to be applied once more.

3.2.3. *Starting Over Again.* In particular, we have to examine the residue class of $\bar{0}$—the open subdisk

$$\mathsf{X}_1 := \{x \in \mathsf{X} \mid 0 < v(t_2(x))\} \subset \mathsf{X}$$

with parameter $t_2 = (t-\xi)/\xi$—and the induced covering

$$\phi_1 :\ \mathsf{Y}_1 := \phi^{-1}(\mathsf{X}_1) \to \mathsf{X}_1$$

thereof. The covering is described by Eq. (3.7), interpreted as an equation over the power series ring $R[\![t_2]\!] = \mathring{\mathcal{O}}_{\mathsf{X}_1}$.

By the above results, h gives rise to a sufficiently precise 3-Taylor expansion of f with respect to the parameter t_2 on X_1. The corresponding critical line segment is $\overline{P_0'P_2} = \overline{(0,3/2)(2,17/12)}$ (in particular, this segment does not start at P_1 and so the considered parameter t_2 is well-suited for our purposes). Already, we can deduce that we will end up with a smooth component of genus one, see the proof of Lem. 2.35. By Prop. 2.33, the radius of the minimal ϕ_1-exhausting disk $\mathsf{D}_1 \subset \mathsf{X}_1$ is

$$\tilde{\rho}_1 = \frac{3/2-17/12}{2} = \frac{1}{24}\,.$$

This is with respect to the parameter t_2, though; with respect to t_1, the radius is

$$\rho_1 = \rho_0 + \tilde{\rho}_1 = \frac{1}{12} + \frac{1}{24} = \frac{1}{8}\,.$$

Note that K_3 does not contain elements of this valuation since the ramification degree of $K_3/\mathbb{Q}_3$ is 36, which is only divisible by 4 and not by 8. Hence, another ramified degree-two-extension M/K_3 is required, and we define

$$M := K_3(\eta \mid \eta^2 = 3^{1/4})\,.$$

Here $3^{1/4}$ denotes any of the fourth roots of 3, which all lie in K_3. (Indeed, substituting $\xi^4 t$ for t in the polynomial t^4-3 and then dividing by ξ^{12} leads to the polynomial $t^4 - 3/\xi^{12}$, which reduces to $\bar{t}^4 - 1 \in \mathbb{F}_9[\bar{t}]$; as $\mathbb{F}_9$ contains all fourth roots of unity, the latter polynomial decomposes into distinct linear factors, and so does, by Hensel's lemma, the initial polynomial.) We assume K to contain M; see Fig. 3.1 for a diagram of the field extensions involved so far.

As an eighth root of 3, the element η has the required valuation $v(\eta) = 1/8$ and can thus be used to obtain a disk parameter

$$t_3 := \frac{t_1}{\eta}$$

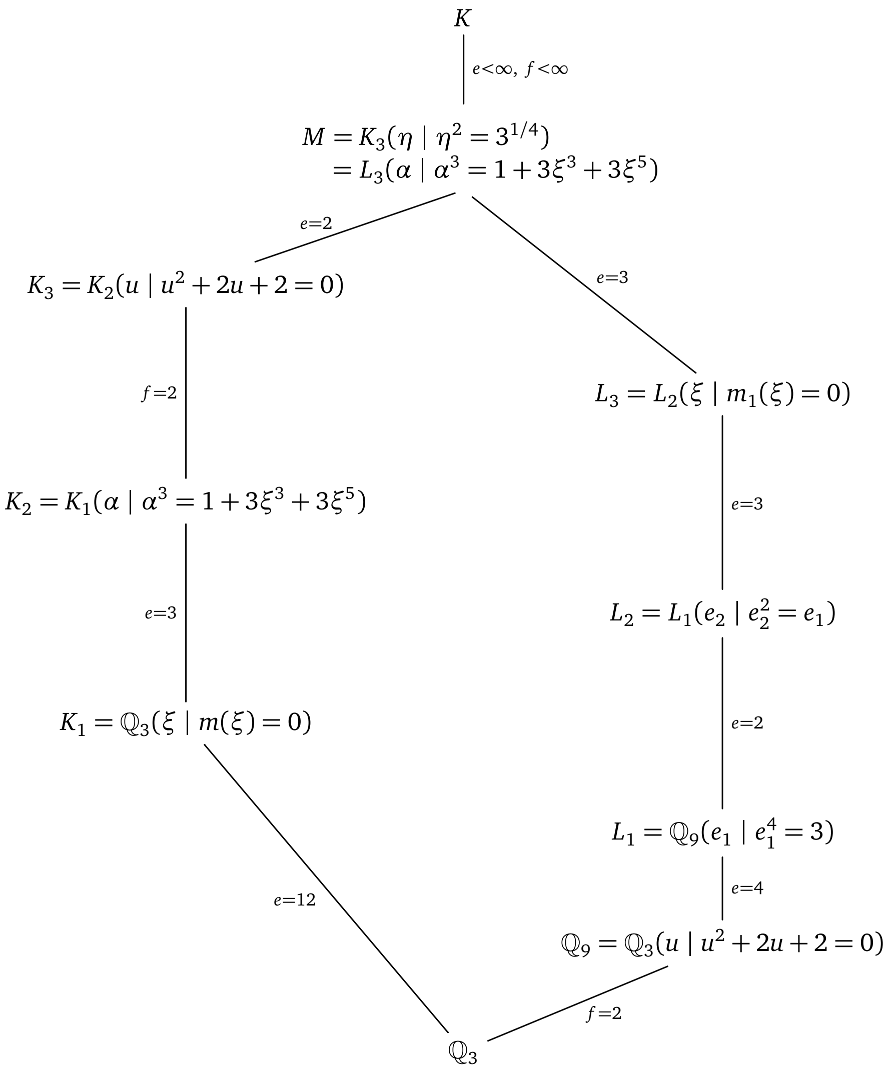

FIGURE 3.1. The tower of fields built up in the course of applying our algorithm is shown on the left hand side; the alternative way of generating the field extension for the study of the monodromy action is shown on the right hand side. $M/\mathbb{Q}_3$ is a Galois extension of degree $[M : \mathbb{Q}_3] = 144$ ($e = 72$, $f = 2$); with respect to the Semistable Reduction Theorem, we are only interested in the ramified part—that is, in the Galois extension $M/\mathbb{Q}_9$.

for the affinoid disk D_1. Note that $\eta^4 \in M$ has valuation $v(\eta^4) = 1/2$, so the variable change

$$w_2 := \frac{y-h}{\eta^4}\,,$$

with h considered as a power series in t_3, leads to an equation well-suited for reduction: as expected, we obtain an Artin–Schreier equation

$$\bar{w}_2^3 + \bar{w}_2 = \bar{u}^5 \bar{t}_3^2 \tag{3.10}$$

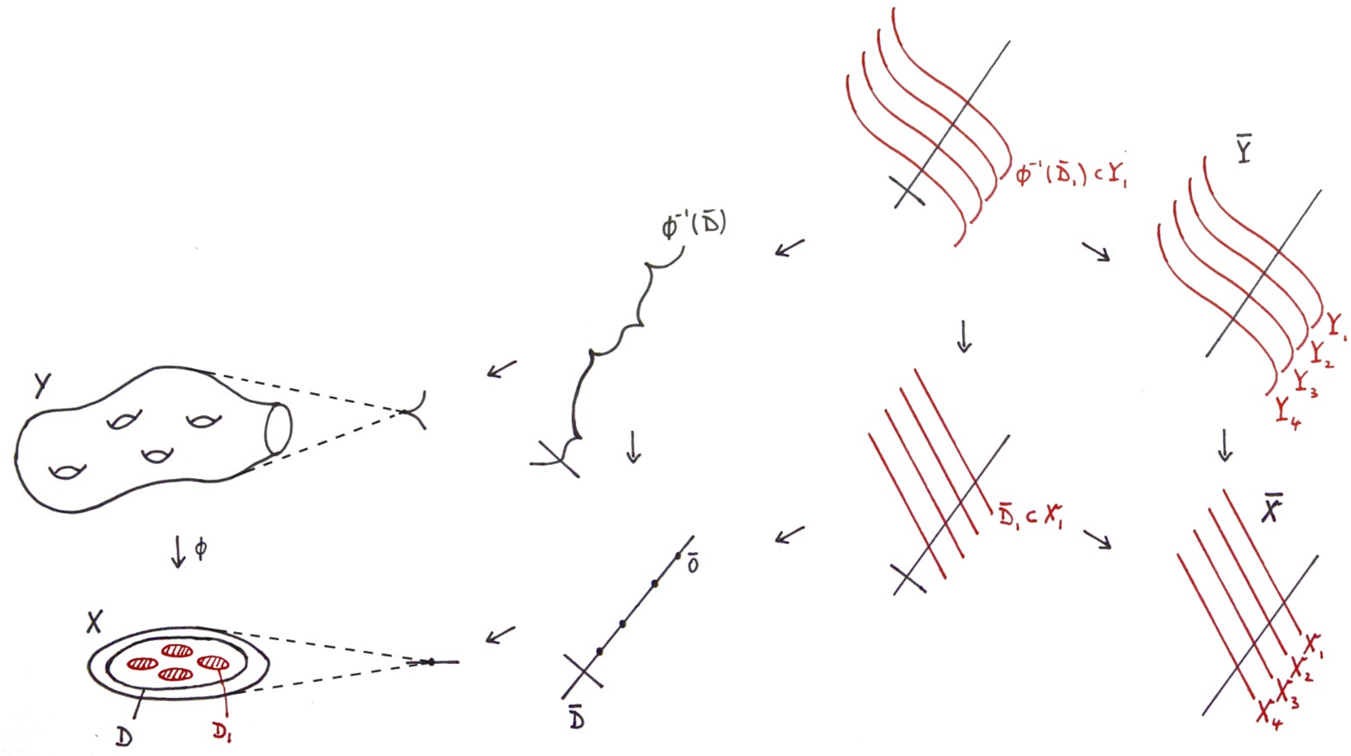

FIGURE 3.2. The models of the 3-cyclic Galois cover $\phi : \mathsf{Y} \to \mathsf{X}$ as produced by our algorithm. The modification induced by the closed disk $\mathsf{D} \subset \mathsf{X}$ splits the original singularity on the special fiber of Y into four singularities less bad. Subsequent modifications, corresponding to smaller disjoint closed disks $\mathsf{D}_1, \dots, \mathsf{D}_4$, lead to four smooth components of genus one. The stable reduction $\bar{Y}$ thus consists of four elliptic components $Y_1, \dots, Y_4$, which are separated by a single rational component and which lie above four rational components $X_1, \dots, X_4 \subset \bar{X}$, respectively.

over $k[\bar{t}_3]$, describing a smooth component Y_1 of genus one lying over a rational component X_1. Here, $\bar{u}$ is a generator for the residue field extension $\mathbb{F}_9/\mathbb{F}_3$; see (3.9).

3.2.4. *Critical Points Remaining.* We claim that the other three critical points behave the same: that is, over each of these points, the algorithm terminates with a smooth component of genus one. Consequently, the stable model of Y is as in the sketch of Fig. 3.2.

Claim 3.1. *With notation as before, we can realize the stable reduction of Y over the field M. The reduction $\bar{Y}$ of the minimal semistable model Y^{ss} consists of four elliptic components $Y_1, \dots, Y_4$, which are separated by a single rational component.*

To justify our claim, we go back to the first improvement step described in Sect. 3.2.2. There, we chose a zero ξ of the Eisenstein polynomial $m(t)$ as a center point for the minimal exhausting disk. Since m is irreducible over $\mathbb{Q}_3$, the Galois group of the polynomial acts transitively on the twelve zeros of m; consequently, those zeros cannot be distinguished and we could have taken any other as well. The Newton polygon of

$m(t+\xi)/t$ reveals that two of the other zeros of m lie close beside the chosen ξ—the valuation of the distance being $1/6$—while the remaining nine zeros are further away—the distance being of valuation $1/10$. It follows that the zeros of m fall into four clusters of three zeros.

Taking into account that, in Sect. 3.2.3, we determined the radius of the closed disk from the final blow-up to be of valuation $1/8$, the four clusters give rise to four disjoint closed disks $\mathsf{D}_i \subset \mathsf{X}$, each of which induces a modification leading to a smooth component Y_i of genus one (lying above a rational component X_i). Note that the four elliptic components Y_i are separated by a rational component (corresponding to the blow-up induced by D) and that the arithmetic genus of $Y_1, \dots, Y_4$ adds up to $g_Y = 4$, the genus of Y. Hence, all relevant components in the semistable reduction of Y have been found. As there are no superfluous rational components, this gives the minimal semistable model Y^{ss} of Y. $\square$

Note that a more formal argument can easily be given by considering the monodromy action on the stable reduction. This will become obvious in the following section, where we determine the monodromy action explicitly.

3.3. **Minimal Semistable Model and Monodromy Action.** Now that we have found a field extension $M/\mathbb{Q}_3$ over which the stable reduction of Y can be realized, as well as the corresponding stable model, we would like to determine the actual monodromy extension. More precisely, we want to show that the ramified part $M/\mathbb{Q}_9$ of the extension $M/\mathbb{Q}_3$ produced by our algorithm is the minimal extension needed for realizing the stable reduction of Y (in the sense of Def. 1.4).

Claim 3.2. *With notation as before, $M/\mathbb{Q}_9$ is a totally ramified Galois extension of degree $[M:\mathbb{Q}_9] = 72$. Its Galois group $\mathrm{Gal}(M/\mathbb{Q}_9)$ acts faithfully on the special fiber $\bar{Y}$ of the stable model Y^{ss} for $Y \otimes_{\mathbb{Q}_9} M$; as a consequence, $M/\mathbb{Q}_9$ corresponds to the monodromy extension of Y.*

To justify the claim, we will establish a number of facts that, taken together, imply our assertions. To start with, we generate the field extension $M/\mathbb{Q}_3$ in a slightly different way and study the behavior of the polynomials involved.

3.3.1. *Computer Algebra System.* We will use the computer algebra system MAGMA [Magma] for performing the necessary calculations. The choice came down to this (instead of one of the open source alternatives) due to MAGMA's rather strong functionality in working with iterated field extensions of high degree, also in the case of p-adic fields. Along the way, we state the source code that was used to obtain the respective results; this allows to reconstruct all details of our calculations.

As p-adic rings are completions, their elements can generally not be represented exactly in a computer algebra system: only a finite amount of data can be stored and processed. Usually, elements are interpreted as power series in a fixed uniformizing element with the infinite expansion truncated at some point—the so-called precision. Effectively, this means to work in a finite quotient of the ring. Since all structural information of a p-adic field (that is, the fraction field of a p-adic ring) is already contained in its ring of integers, field elements can be represented as the product of a unit (known to the chosen fixed precision) with a suitable (negative or positive) power of the uniformizer. MAGMA uses the big-O-notation in its output to indicate the precision up to which a result is known. Confer the corresponding implementation documentation [Magma].

Because of the limited precision we have to work with, all results obtained are a priori not exact and have to be taken with some care. Due to Krasner's lemma [BGR84, Sect. 3.4.2], however, one knows that the obtained results reflect the exact situation, provided the finite precision for the local field was chosen high enough. For example, if a polynomial is recognized to be irreducible up to a certain finite precision (depending on the zeros of the polynomial), then we know that it really *is* irreducible over the local field; or, if a field extension is generated by the zero of a polynomial that itself is known only up to a certain precision, then the extension generated coincides with the extension generated by a zero of the exact polynomial, if only the precision is high enough.

Remark 3.3. In the above sense, all of the following results obtained with MAGMA's p-adic functionality are only true 'modulo showing that the precision is chosen high enough'. This could be overcome by tracking the field elements involved and determining in every step how high the precision needs to be; we dispense with that. However, the provided source code can serve as a sound basis for those calculations.

The following creates a 3-adic field with precision 12, which serves as the base field we start with.

```
Q3       := pAdicField(3,12);
```

3.3.2. *Field Tower.* To generate the field M, we proceed in several steps. We begin with the maximal unramified subextension of $M/\mathbb{Q}_3$. It is the unique unramified extension

$$\mathbb{Q}_9 := \mathbb{Q}_3(u \mid u^2 + 2u + 2 = 0)$$

of degree two over $\mathbb{Q}_3$.

```
Q9<u>    := UnramifiedExtension(Q3,2);
```

The next step is to take care of the tamely ramified part. We set

$$L_1 := \mathbb{Q}_9(e_1 \mid e_1^4 = 3) \quad \text{and} \quad L_2 := L_1(e_2 \mid e_2^2 = e_1)\,.$$

The tamely ramified extension $L_2/\mathbb{Q}_9$ is Galois of degree $[L_2 : \mathbb{Q}_9] = 8$, as L_2 contains all eighth roots of unity (since the residue field $\mathbb{F}_9$ does).

```
A<t>     := PolynomialRing(Q9);
L1<e1>   := TotallyRamifiedExtension(Q9,t^4-3);
A<t>     := PolynomialRing(L1);
L2<e2>   := TotallyRamifiedExtension(L1,t^2-e1);
```

The polynomial $m(t)$ from (3.4), which is irreducible over $\mathbb{Q}_9$ and which gives the center ξ of the minimal exhausting disks D and D_1 (see Sects. 3.2.2 and 3.2.3), splits over L_1 resp. L_2 into four factors,

$$m = m_1 m_2 m_3 m_4\,,$$

each factor being of degree three.

```
m        := 3+30*t^2+18*t^3+198*t^5+180*t^7+189*t^8\
            +342*t^10+145*t^12;
facm     := Factorization(m);
```

Observe that the expressions for the involved polynomials and field extensions get more and more complex; for instance, the first factor of m is as follows:

$$\begin{aligned} m_1 = t^3 &+ (((169839u - 191763)e_1^2 + (19098u + 19098)e_1 - 127799u)e_1 \\ &+ O(e_1^{48}))t^2 + ((6882ue_1^3 + 25586e_1^2 + (-14892u - 29784)e_1 \\ &+ (28195u + 28195))e_1^6 + O(e_1^{48}))t + (82152e_1^3 + (-125639u \\ &- 251278)e_1^2 + (82218u + 82218)e_1 - 181808u)e_1 + O(e_1^{48}) \,. \end{aligned}$$

Note that the 4-cyclic Galois group $\mathrm{Gal}(L_1/\mathbb{Q}_9)$ acts transitively on the four factors $m_1, \ldots, m_4$. Also, recall that we saw the zeros of m fall into four clusters of three zeros (Sect. 3.2.4)—these precisely correspond to the four factors of m over L_1, which in turn correspond to the four distinct rational components $X_1, \ldots, X_4 \subset \bar{X}$. It becomes clear again that the stable model Y^{ss} of Y consists of four elliptic components lying over four rational components: having found the part $Y_1 \to X_1$ of the stable reduction, the remaining parts of the stable model are determined by the monodromy action, which, in particular, transitively permutes the factors $m_1, \ldots, m_4$ of m, thus giving rise to the isomorphic parts $Y_2 \to X_2, \ldots, Y_4 \to \mathrm{X}_4$. For later usage, we note:

Fact 3.4. The cyclic Galois group $\mathrm{Gal}(L_1/\mathbb{Q}_9) \cong \mathbb{Z}/4$ permutes the factors $m_1, \ldots, m_4$ of m; as a consequence, the monodromy action on the reduction of Y permutes the four isomorphic parts $Y_1 \to X_1, \ldots, Y_4 \to X_4$. □

We continue with generating the field extension $M/\mathbb{Q}_9$. The next step is to choose one of the factors of m over L_2, say m_1, and to adjoin one of its roots ξ to L_2. This gives the field

$$L_3 := L_2(\xi \mid m_1(\xi) = 0) \,,$$

which is totally ramified of degree three over L_2. To do so in MAGMA, we cannot directly use the polynomial m_1. This is because MAGMA requires totally ramified field extensions to be defined by Eisenstein polynomials, whereas m_1 is not of Eisenstein type. However, by an appropriate substitution of the variable t, we can transform m_1 into an Eisenstein polynomial $\tilde{m}_1$ suitable for us. The idea behind this is similar to that of p-Taylor expansion: what we want is a polynomial where the corresponding Newton polygon consists of only one line segment, with the absolute value of the slope being the valuation of the uniformizing element we are looking for. By a combination of shifting, scaling, and inverting of the variable t, we can transform the original Newton polygon into the desired form.

More specifically, in the above situation, we first substitute e_2/t for t, then multiply by t^3, and finally divide by the leading coefficient.

```
A<t>     := PolynomialRing(L2);
m1       := A!facm[1][1];
m1t      := Evaluate(m1,t*e2);
m1t      := Coefficients(m1t)[4]+Coefficients(m1t)[3]*t\
            + Coefficients(m1t)[2]*t^2+Coefficients(m1t)[1]*t^3;
m1t      := m1t/Coefficients(m1t)[4];
[Valuation(CC)/AbsoluteRamificationDegree(L2) : CC in \
            Coefficients(m1t)];
```

The resulting polynomial $\tilde{m}$ is indeed Eisenstein—its coefficients being of valuation

$$\left[\frac{1}{8}, \frac{2}{8}, \frac{11}{8}, 0\right]$$

—and as such can be used in MAGMA to generate the extension $L_3' = L_2(\tilde{\xi})$.[10] As m_1 splits into linear factors over L_3', we have that $L_3' = L_3 = L_2(\xi)$. We also observe that m_3 splits into linear factors as well, whereas m_2 and m_4 remain irreducible.

```
m1t      := A![Expand(CC) : CC in Coefficients(m1t)];
L3<xit> := TotallyRamifiedExtension(L2,m1t);
A<t>     := PolynomialRing(L3);
facm1    := Factorization(A!m1);
xi       := -Coefficients(facm1[1][1])[1];
Factorization(A!facm[3][1]);
IsIrreducible(A!facm[2][1]);
IsIrreducible(A!facm[4][1]);
```

Recall the situation from Sect. 3.2.2: after adjoining ξ, we had to adjoin a root α of

$$n(t) := t^3 - (1 + 3\xi^3 + 3\xi^5)$$

to get the totally ramified degree-three-extension K_2/K_1. Again, since n is not Eisenstein, we cannot use it directly in MAGMA; instead, we have to transform it into a suitable Eisenstein polynomial $\tilde{n}$.

```
n        := t^3-(1+3*xi^3+3*xi^5);
nt       := Evaluate(n,t*xi^5+1)/(3*xi^3);
nt       := Evaluate(nt,xit*t+Coefficients(nt)[1]);
nt       := nt/Coefficients(nt)[4];
[Valuation(CC)/AbsoluteRamificationDegree(L3) : CC in \
           Coefficients(nt)];
```

With the coefficients having valuation

$$\left[\frac{1}{24}, \frac{2}{24}, \frac{13}{24}, 0\right] ,$$

the resulting polynomial $\tilde{n}$ is indeed Eisenstein and can be used in MAGMA to generate a field extension $M' = L_3(\tilde{\alpha})$.[11] As n splits over M' into linear factors, we have

$$M' = L_3(\alpha \mid \alpha^3 = 1 + 3\xi^3 + 3\xi^5) .$$

Over the field M', the polynomials m_2 and m_4 also split into linear factors.

```
nt       := A![Expand(CC) : CC in Coefficients(nt)];
M<at>    := TotallyRamifiedExtension(L3,nt);
A<t>     := PolynomialRing(M);
facn     := Factorization(A!n);
a        := -Coefficients(facn[1][1])[1];
Factorization(A!facm[2][1]);
Factorization(A!facm[4][1]);
```

[10] This is one of the places where we assume that our calculations are done with high enough precision: the resulting $\tilde{m}_1$ is not known to the full precision of the base ring and as such cannot be used in MAGMA for generating an extension field; one has to assure MAGMA that this polynomial *is* known to a sufficiently high precision and it shall work with the polynomial *as is* (this is done with the 'Expand' command).

[11] Here, the analog of Footnote 10 also applies.

Taking the degree of the respective extensions into account, we deduce that M' is the splitting field over $\mathbb{Q}_9$ of the separable polynomials t^8-3 and $m(t)$; the extension $M'/\mathbb{Q}_9$ is hence Galois and totally ramified of degree 72. But it also follows that M' coincides with M: as seen above, the polynomials used to generate M (Sects. 3.2.1–3.2.3) all split over M' into linear factors, and both M and M' have degree 72 over $\mathbb{Q}_9$. We have established the following fact, proving the first part of Claim 3.2.

Fact 3.5. $M/\mathbb{Q}_9$ is a totally ramified Galois extension of degree $[M:\mathbb{Q}_9]=72$. □

3.3.3. *Monodromy Action I.* It remains to show that $\Gamma := \mathrm{Gal}(M/\mathbb{Q}_9)$ acts faithfully on the reduction $\bar{Y}$ of Y. We will again proceed in several steps.

As above, let the stable model of Y be denoted by Y^{ss} and the induced semistable model of X (resulting as the quotient of Y^{ss} under the action of the p-cyclic Galois group of the covering Φ) by X^{ss}. As seen in Claim 3.1 and its proof, $\bar{Y}$ consists of four elliptic components $Y_1,\dots,Y_4$, which are separated by a single rational component; also, the components Y_i lie above rational components X_i of $\bar{X}$ (which again are separated by a single rational component). In the following, we will study the action of Γ on $\bar{Y}\to\bar{X}$ and the action of certain subgroups on the single parts $Y_i\to X_i$.

We know from Fact 3.4 that Γ transitively permutes the components $X_1,\dots,X_4\subset\bar{X}$ and that the normal subgroup $\mathrm{Gal}(M/L_1)$ fixes these (the permutation representation on the four components is hence via the quotient $\mathrm{Gal}(L_1/\mathbb{Q}_9)\cong\mathbb{Z}/4$). Consequently, we can study the action of $\mathrm{Gal}(M/L_1)$ on a single component, say on X_1. This component corresponds to the modification with center ξ and is described in terms of the parameter $t_3=(t-\xi)/\eta$. We would like to see what effect the Galois automorphisms from $\mathrm{Gal}(M/L_1)$ have thereon.

Each $\sigma\in\mathrm{Gal}(M/L_1)$ fixes two points of the rational component X_1 (namely, the point where ξ reduces to and the point at infinity) and can therefore only act via multiplication with a root of unity contained in $\mathbb{F}_9$. As $\mathrm{Gal}(M/L_1)$ is of order $18=2\cdot 9$, this root of unity can only be 1 (corresponding to the identity) or -1 (giving a non-trivial action). We calculate

$$\sigma(t_3)=\frac{t-\sigma(\xi)}{\sigma(\eta)}=\frac{\eta}{\sigma(\eta)}\left(t_3+\frac{\xi-\sigma(\xi)}{\eta}\right). \tag{3.11}$$

Since the distance between the zeros of m_1 is 1/6—whereas $\nu(\eta)=1/8$—we obtain in reduction

$$\sigma(\bar{t}_3)=\overline{\frac{\eta}{\sigma(\eta)}}\,\bar{t}_3\,.$$

This depends only on the 2-cyclic quotient $\mathrm{Gal}(L_2/L_1)=\mathrm{Gal}(M/L_1)/\mathrm{Gal}(M/L_2)$ (with its non-trivial element acting via multiplication by -1); the kernel of the action of $\mathrm{Gal}(M/L_1)$ on X_1 is thus $P=\mathrm{Gal}(M/L_2)$. The kernel with respect to one of the other components $X_2,\dots,X_4$ is a conjugate of the normal P, that is, also P. Taken together with Fact 3.4, we obtain

Fact 3.6. The action of $\Gamma=\mathrm{Gal}(M/\mathbb{Q}_9)$ on $\bar{X}$ has kernel $P=\mathrm{Gal}(M/L_2)$. □

To see that Γ acts faithfully on the reduction of Y, it is thus enough to show that the wild ramification subgroup $P=\mathrm{Gal}(M/L_2)$ acts faithfully on $\bar{Y}$. A first step towards this aim is to establish the following fact.

Fact 3.7. P acts non-trivially on Y_1; that is, the kernel P_1 of $P\to\mathrm{Aut}(Y_1)$ is of order $|P_1|=3$.

Recall that Y_1 is described by Eq. (3.10). To determine the effect of an element $\sigma \in P$ on Y_1, we have to see what happens with $\bar{w}_2$ and $\bar{t}_3$. We have already seen that $\bar{t}_3$ is fixed by all elements of P. Concerning the reduction $\bar{w}_2$ of $w_2 = (y-h)/\eta^2$ (with $h = \alpha + \beta\eta t_3$), we calculate

$$\sigma(w_2) = \frac{y - \sigma(h)}{\eta^2}$$

and, accordingly,

$$\sigma(w_2) - w_2 = \frac{h - \sigma(h)}{\eta^2} = \frac{(\alpha - \sigma(\alpha)) + \eta(\beta t_3 - \sigma(\beta)\sigma(t_3))}{\eta^2}\,.$$

The reduction of the latter expression describes the effect of $\sigma \in P$ on $\bar{w}_2$; though $\bar{t}_3$ itself does not change under these automorphisms, the action on t_3 (as described in (3.11)) must very well be taken into account.

3.3.4. *Interlude: Calculating Galois Actions.* To study the action of Galois automorphisms on field elements and on polynomials with the help of MAGMA, we have written a number of procedures based on facts from basic field theory.

Recall that the E_1-homomorphisms of a simple field extension $E_2 := E_1(a_1)$ into an algebraic closure E_2^{ac} are in one-to-one correspondence with the distinct zeros of the minimal polynomial $p_{a_1}(t) \in E_1[t]$ in E_2^{ac}; see [Bos01, Lem. 3.4.8]. More precisely, those homomorphisms are determined by the image of a_1, and precisely the zeros of $p_{a_1}(t)$ can be taken as image elements. This is also applicable for towers $E_n := E_{n-1}(a_{n-1}), \ldots, E_2 := E_1(a_1)$ of simple field extensions: iterating through each extension, one notes where the respective generator a_i can be mapped to. Each homomorphism is then represented by a list of field elements $[\tilde{a}_1, \ldots, \tilde{a}_n]$ and the action on an arbitrary field element a can be calculated by first representing a in terms of the generators $a_1, \ldots, a_n$ and then substituting $\tilde{a}_1, \ldots, \tilde{a}_n$ for these.

This is what the recursively defined procedures *FindHoms* and *ApplyHom* do. The former generates a list of all E_1-homomorphisms of E_2 into a sufficiently large extension field by iterating through the simple extensions that were used to generate the field E_2 over E_1; each homomorphism gets represented as a list of suitable field elements. The latter procedure can be used to apply a homomorphism (given in the above list form) by iterating through the subextensions with its generators, and substituting in each step the respective list element.

```
function ApplyHom(f,homs)
    if IsEmpty(homs) then
        result := f;
    else
        hom := homs[#homs];
        C := Coefficients(f);
        result := 0;
        for i in [0..#C-1] do
            result := result+ApplyHom(C[i+1],Prune(homs))*hom^i;
        end for;
    end if;
    return result;
end function;
```

```
function FindHoms(homs,E1,E2,A)
    result := [**];
    if not E1 eq BaseField(E2) then
        for psi in FindHoms(homs,E1,BaseField(E2),A) do
            result := result cat FindHoms(psi,BaseField(E2),E2,A);
        end for;
    else
        pol := DefiningPolynomial(E2);
        fac := Factorization(A!ApplyHom(pol,homs cat \
                  [*SetToIndexedSet(Generators(A))[1]*]));
        for i in [1..#fac] do
            result := Append(result,Append(homs,\
                        -Coefficients(fac[i][1])[1]));
        end for;
    end if;
    return result;
end function;
```

The procedure ApplyHom can also be used to act on polynomials (by simultaneously acting on all coefficients). For this, one has to extend the list representing a given homomorphism with an additional entry for the generator of the polynomial ring. This is done with the following procedure.

```
function ExtendHomList(homs,A)
    for i in [1..#homs] do
        Append(~homs[i],SetToIndexedSet(Generators(A))[1]);
    end for;
    return homs;
end function;
```

3.3.5. *Monodromy Action II.* We will use the procedures described in Sect. 3.3.4 to calculate the differences $\sigma(w_2) - w_2$ for all automorphisms $\sigma \in P$; this will show which of the automorphisms act trivially on the component $Y_1 \subset \bar{Y}$.

First, we generate a list of all automorphisms from P.

```
homs    := FindHoms([**],L2,M,A);
homs2   := ExtendHomList(homs,A);
```

We denote the nine list elements by $\sigma_1, \ldots, \sigma_9$; these correspond to the elements of $P = \mathrm{Gal}(M/L_2)$. We then apply these automorphisms and determine the differences $\Delta_i := w_2 - \sigma_i(w_2)$ resp. the corresponding valuations $v(\Delta_i)$ and reductions $\bar{\Delta}_i$.

```
f       := 1+3*t^3+3*t^5;
f2      := Evaluate(f,e2*t+xi);
b       := (9*xi^2+15*xi^4)/(3*a^2);
h2      := Evaluate(a+b*t,e2*t);

k<u>,phi := ResidueClassField(RingOfIntegers(M));
for i in [1..#homs] do
    sigmaxi := ApplyHom(M!xi,homs[i]);
    sigmah2 := Evaluate(ApplyHom(h2,homs2[i]),t+(xi-sigmaxi)/e2);
    delta   := (sigmah2-h2)/e1^2;
```

$\sigma_i \in P$	$v(\Delta_i)$	$\bar{\Delta}_i \in \mathbb{F}_9$
σ_1	∞[1]	0
σ_2	0	u^2
σ_3	0	u^6
σ_4	0	u^2
σ_5	1/24	0
σ_6	0	u^6
σ_7	0	u^2
σ_8	0	u^6
σ_9	1/24	0

[1] The actual value is $v(\sigma_1) = 275/24$; with respect to the finite precision we are bound to, this corresponds to an arbitrary small element.

FIGURE 3.3. The effect of the automorphisms σ_i from the wild ramification subgroup $P = \mathrm{Gal}(M/L_2)$ on the elliptic component Y_1, stated in terms of the differences $\Delta_i = w_2 - \sigma_i(w_2)$.

```
    print "Hom  #",i,":  min =",Minimum([Valuation(CC)/\
        AbsoluteRamificationDegree(M) : CC in Coefficients(delta)]);
    print "     \\bar{delta} =",PolynomialRing(k)![phi(CC) : \
        CC in Coefficients(delta)];
end for;
```

The essentials of our computations are presented in Fig. 3.3. We deduce that precisely the automorphisms σ_1, σ_5, and σ_9 act trivially on Y_1, giving a kernel P_1 of order three. This justifies the assertion from Fact 3.7. □

The kernel P_i of the action of $P = \mathrm{Gal}(M/L_2)$ on one of the other components $Y_i \subset \bar{Y}$ is given by $P_i = \tau P_1 \tau^{-1}$, where $\tau \in \Gamma = \mathrm{Gal}(M/\mathbb{Q}_9)$ is a Galois automorphism with $\tau(Y_1) = Y_i$ (such automorphisms exist because Γ transitively permutes the parts $Y_i \to X_i$, see Fact 3.4). The crucial point is to see that P_1 is not invariant under conjugation with elements of Γ. If this were to be true, the kernel with respect to the action on the whole of $\bar{Y}$—namely, the intersection of the kernels $P_1, \dots, P_4$—would necessarily be trivial.

To this end, we examine the permutation representation of P_1 on the twelve zeros $\xi_1, \dots, \xi_{12}$ of $m(t)$, which we enumerate with respect to the four clusters they fall into: $\xi_1 := \xi, \xi_2, \xi_3$ shall be the zeros of m_1, the zeros of m_2 shall be ξ_4, ξ_5, ξ_6, and so on. We then apply the three automorphisms from P_1 to $\xi_1, \dots, \xi_{12}$ and determine in which way the twelve elements get permuted.

```
xis     := [**];
for i in [1..4] do
    facmi := Factorization(A!facm[i][1]);
    for j in [1..#facmi] do
        Append(~xis,-Coefficients(facmi[j][1])[1]);
    end for;
end for;
```

```
P1       := [*homs[1],homs[5],homs[9]*];
for i in [1..#P1] do
    perm := [**];
    for j in [1..#xis] do
        sigmaxi := ApplyHom(xis[j],P1[i]);
        max,k := Maximum([sigmaxi-CC : CC in xis]);
        Append(~perm,k);
    end for;
    print "Action of element #",i,"from P1 on [1,..,12]:",perm;
end for;
```

This leads to the following result.

Fact 3.8. With notation as above, the permutation representation of $P_1 \lhd P$ on the twelve zeros of $m(t)$ is of the form

$$\sigma_1 = \mathrm{id}\,, \quad \sigma_5 = (1\ 3\ 2)\,(7\ 8\ 9)\,, \quad \sigma_9 = (1\ 2\ 3)\,(7\ 9\ 8)\,.$$

In particular, P_1 is not invariant under conjugation with elements from $\mathrm{Gal}(M/\mathbb{Q}_9)$. □

As a consequence of the established facts 3.5–3.8, Claim 3.2 is finally justified: as $\Gamma = \mathrm{Gal}(M/\mathbb{Q}_9)$ is seen to act faithfully on $\bar{Y}$, the totally ramified Galois extension $M/\mathbb{Q}_9$ of degree 72 indeed corresponds to the monodromy extension of Y. □

4. Prime-Cyclic Étale Galois Covers of Open Annuli

In Sect. 2, we were able to construct the minimal exhausting disk for prime-cyclic étale covers of the open unit disk. The ramified situation will naturally lead us to the study of coverings of open annuli, which is another important ingredient in our proof of the Semistable Reduction Theorem; namely, for being able to reduce the case of a general solvable Galois group to the prime-cyclic case, coverings of open annuli have to be handled.

4.1. **Ramified Covers of the Open Unit Disk.** Let $\phi : \mathsf{Y} \to \mathsf{X}$ be a p-cyclic Galois covering of the open disk X, and assume ϕ to be ramified in the finitely many points $x_1, \ldots, x_n \in \mathsf{X}$. Choosing an appropriate parameter t for the ring $A := \mathring{\mathscr{O}}_{\mathsf{X}}$ of zero-bounded analytic functions on X, we can assume $x_1 = 0$.

Again, our goal is to construct the minimal ϕ-exhausting disk D, which leads to a modification improving the model of Y. Instead of constructing this disk at once, we introduce intermediate steps by considering the *minimal exhausting disk $\tilde{\mathsf{D}}$ that contains all ramification points*—that is, we show that the set of all ϕ-exhausting disks containing all ramification points $x_1, \ldots, x_n$ has a unique minimum, and we give an explicit construction of this minimal element. In contrast to the étale situation, finding a suitable center point for $\tilde{\mathsf{D}}$ is hence no issue: we can take $x_1 = 0$. Note that $\tilde{\mathsf{D}}$ necessarily contains

$$\mathsf{D}_{\mathrm{ram}} := \{x \in \mathsf{X} \mid \min_{i=1,\ldots,n} v(x_i) \le v(x)\}\,.$$

For the moment, suppose we can find and construct $\tilde{\mathsf{D}}$. As $\tilde{\mathsf{D}} \supset \mathsf{D}$, two cases can occur. In the first case, $\tilde{\mathsf{D}} = \mathsf{D}$ holds true and we have found the minimal ϕ-exhausting disk we are looking for. In the second case, $\tilde{\mathsf{D}} \supsetneq \mathsf{D}$; note that then $\tilde{\mathsf{D}} = \mathsf{D}_{\mathrm{ram}}$ holds true. Though in this case, $\tilde{\mathsf{D}}$ does not correspond to an improving modification in the sense of Sect. 1.2.3, it nevertheless improves the situation: the modification induced by $\tilde{\mathsf{D}}$

separates at least some of the original ramification points in reduction. Consequently, the residue classes of critical points will contain strictly fewer ramification points than before. As this can happen only a finite number of times, we will finally end up with an improving modification, as desired.

The above shows that it suffices to determine the minimal ϕ-exhausting disk containing all ramification points of the covering. This is equivalent to determining the maximal separating boundary domain for the *étale* covering $\phi^{-1}(\mathsf{A}) \to \mathsf{A}$ of the open annulus $\mathsf{A} := \mathsf{X} \setminus \mathsf{D}_{\text{ram}}$. We note:

Remark 4.1. The ramified disk case is settled as soon as one understands the slightly more general[12] situation of p-cyclic étale Galois covers of open annuli; the same arguments also show that this is enough to settle the case of ramified p-cyclic coverings of open annuli. Note that the reduction step might require to replace K by a sufficiently large finite extension to have the involved models be permanent (see Sect. 1.2.2). Though the situation is more complex than in the disk case, our algorithmic procedure will automatically produce a suitable finite extension, with the consequence that the reduction step is all right also with respect to practical applications; see Rem. 4.20.

4.2. **Setting.** We present the situation the rest of Sect. 4 is devoted to, and we fix some notation, which will be in force during the remainder of this section. The base field K is still assumed to be of characteristic $(0, p)$ and to contain a primitive pth root of unity.

4.2.1. *Étale Coverings of Open Annuli.* In the following, X shall denote an open annulus of thickness $\epsilon \in v(K^\times)$ and $\phi : \mathsf{Y} \to \mathsf{X}$ a p-cyclic étale Galois covering thereof. We wish to prove the analog of Thm. 2.1 for coverings of open annuli.

Theorem 4.2. *Let $\phi : \mathsf{Y} \to \mathsf{X}$ be a p-cyclic étale Galois cover of the open annulus X. Fix a boundary point $\eta_1 \in \partial \mathsf{X}$. Then the set of ϕ-separating boundary domains with regard to η_1 has a unique maximum with respect to inclusion, and this maximum can explicitly be constructed.*

Denote by $B := \mathring{\mathcal{O}}_{\mathsf{Y}}$ and $A := \mathring{\mathcal{O}}_{\mathsf{X}}$ the respective rings of zero-bounded analytic functions. We will construct the maximal separating boundary domain with respect to the boundary point $\eta_1 \in \partial \mathsf{X}$; the boundary point distinct from η_1 will be denoted $\eta_2 \in \partial \mathsf{X}$. Let $t \in A$ be a parameter with respect to η_1. As explained in Sect. 1.5, this gives an isomorphism

$$\mathsf{X} \cong \{x \in \mathbb{A}^1_K \mid 0 < v(t(x)) < \epsilon\} .$$

In this sense, η_1 is conceived as the 'outer' boundary and η_2 as the 'inner' boundary of the annulus X. Also, denoting by $a \in K$ any element with valuation $v(a) = \epsilon$, we have

$$A = R[\![s, t \mid st = a]\!] .$$

Taking $s = at^{-1}$ into account, elements $f \in A$ can thus be interpreted as Laurent series in t,

$$f = \sum_{i=-\infty}^{\infty} a_i t^i ;$$

[12] Not every étale Galois cover of an open annulus is the restriction of a Galois cover of an open disk; for example, consider the étale covering of $\{x \mid 0 < v(t(x)) < 1\}$ given by $y^p = \sum_{i=-\infty}^{\infty} p^{|i/p|} t^i$.

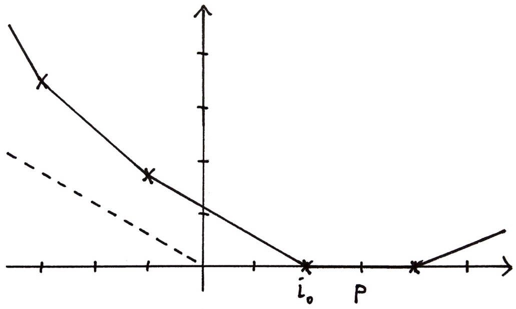

FIGURE 4.1. When representing a zero-bounded analytic function on an open annulus X (of thickness ϵ) as a formal Laurent series $\sum_{i=-\infty}^{\infty} a_i t^i$ in a parameter t, all points $P_i = (i, v(a_i))$ lie in a sector of the upper half plane, which is bounded from below by a line through the origin of slope $-\epsilon$. When the function has no zeros within X, only non-negative or negative enough slopes occur in the associated Newton polygon.

here the coefficients $a_i \in R$ have to satisfy the growth condition

$$v(a_i) \geq \max(-i\epsilon, 0)\,, \tag{4.1}$$

involving the thickness ϵ of the annulus. Figure 4.1 visualizes this condition.

As explained in Sect. 1.4.2, the covering ϕ is given by the ring extension B/A, where B is determined as the integral closure of A in the p-cyclic extension of fraction fields $K(\mathsf{Y})/K(\mathsf{X})$. As K is assumed to contain a primitive pth root of unity, the extension is given by an irreducible Kummer equation

$$y^p = f\,, \quad \text{with } f \in A\,. \tag{4.2}$$

In contrast to the situation from Sect. 2.1, we can in general not assume that $f \in A$ is a unit: there are zero-bounded analytic functions without zeros on X, which still are not invertible as elements of A (for example, $t \in A$). Nevertheless, due to the assumption of ϕ being étale, reasoning as in loc.cit. shows that we can still assume f to have no zeros on X. The Newton polygon of f will then only involve line segments with non-negative slopes (corresponding to zeros beyond the outer boundary η_1) or slopes smaller or equal to $-\epsilon$ (corresponding to zeros beyond the inner boundary η_2); see Fig. 4.1 for an illustration of the situation.

With the usual notation for the rank-two-valuations associated to the boundary points of X, we set $(\alpha_0, i_0) := v_{\eta_1}(f)$. As always, by Conv. 1.6, we are free to assume that K contains a pth root $a_{i_0}^{1/p}$ of a_{i_0}. We can then replace y by $a_{i_0}^{1/p} y t^{\lfloor i_0/p \rfloor}$ in Eq. (4.2) and consider the new equation $y^p = f/a_{i_0} t^{i_0}$. Observe that the function on the right hand side is still an element of A, as the growth condition (4.1) remains satisfied due to the Newton polygon of f being of the 'étale' form described above. We will therefore assume from now on that

$$v_{\eta_1}(f) = (0, i_0)\,, \quad \text{with } 0 \leq i_0 < p\,. \tag{4.3}$$

Only the case $i_0 = 0$ is interesting and requires further investigation because X itself is readily recognized to be separating when $i_0 \in \{1, \ldots, p-1\}$: Indeed, in this case, Eq. (4.2) leads to an irreducible and purely inseparable equation both over the outer boundary and over the inner boundary, as in reduction we get

$$\bar{y}^p = \bar{t}^{i_0} + \text{terms of higher order} \quad \text{and} \quad \bar{y}^p = \bar{s}^{-i_0} + \text{terms of higher order}\,,$$

respectively.[13] As a consequence, above each of the two boundary points of X, there is only one boundary point of Y; moreover, $y \in B$ satisfies conditions (1) and (2) of Lem. 1.31 since $\mathrm{Norm}(y) = f = t^{i_0}u \in A$, with $u \in A$ a unit and with i_0 prime to p. We conclude that X is separating.

We will thus assume $i_0 = 0$ from now on; therewith, we have reduced our studies to the case where $f \in A$ is a unit.

4.2.2. *Number of Boundary Points Above.* Recall Cor. 2.4, which states that, in the case of p-cyclic étale coverings of the disk, there is only one boundary point lying above the boundary of the disk. This is no longer true when studying p-cylic coverings of open annuli: both above η_1 and $\eta_2 \in \partial\mathsf{X}$, there might be lying p boundary points of Y.

Lemma 4.3. *Let A be an open annulus of thickness $v(\pi)$ (with π denoting a uniformizer of R) and consider a p-cyclic étale Galois cover thereof. Then at least one of the two boundary points of A does not split in the covering.*

Proof. In the situation of the lemma, the ring $\mathring{\mathcal{O}}_{\mathsf{A}}$ of zero-bounded analytic functions on A is a regular local ring. By contradiction, suppose that the covering decomposes over both boundaries of A. As a consequence of the theorem on purity of branch loci, the covering then has to decompose over the whole of A—contradicting the fact that open analytic curves are absolutely connected, see Rem. 1.8. □

With the above lemma, we can reduce to a simpler situation concerning ϕ; namely, we can assume the covering to not split over at least one of the boundaries of X. Indeed, suppose that both boundary points $\eta_1, \eta_2 \in \partial\mathsf{X}$ decompose in ϕ. We then subdivide the annulus X of thickness ϵ into a chain of subannuli

$$\mathsf{A}_i := \{x \in \mathsf{X} \mid i v(\pi) < v(t(x)) < (i+1)v(\pi)\}\,, \quad \text{for } i = 0, \ldots, \epsilon/v(\pi) - 1\,,$$

and consider the corresponding restricted coverings $\phi_i : \phi^{-1}(\mathsf{A}_i) \to \mathsf{A}_i$. Note that each annulus A_i is of thickness $v(\pi)$. By Lem. 4.3 and its proof, the covering ϕ_0 is either trivial or does not decompose over the inner boundary of A_0. In the former case, A_0 is contained in the maximal ϕ-separating boundary domain $\mathsf{A} \subset \mathsf{X}$ (with respect to η_1), and we continue and examine A_1, A_2, and so forth. As the open analytic curve Y is not a disjoint union of p open annuli, we eventually end up with an A_i over which the restricted covering ϕ_i does not decompose. Then the union of $\mathsf{A}_0, \ldots, \mathsf{A}_{i-1}$, and of the maximal ϕ_i-separating boundary domain (with respect to the outer boundary) constitutes the maximal ϕ-separating boundary domain $\mathsf{A} \subset \mathsf{X}$.

By the results that we are to establish in Sect. 4.5.2, we will be able to recognize (within finitely many steps) whether the covering decomposes over a given boundary. With respect to determining the maximal separating boundary domain of ϕ, we can hence assume without loss of generality that over at least one boundary point of X, there is only one boundary point of Y.

[13] For being able to write down the second equation, K needs to contain an element with valuation $v_{\mathfrak{p}_2}(f)/p$ (where $v_{\mathfrak{p}_2}$ denotes the discrete valuation associated to $\eta_2 \in \partial\mathsf{X}$); as always, we may assume K to do so.

Analog to the disk case, we speak of the geometric situation; that is,

$$\text{the Laurent series } f \in A \text{ is a unit of } A \text{ and not a } p\text{th power in } A\,, \text{ and over at least one of the two boundaries of } \mathbb{X}, \text{ the covering given by } y^p = f \text{ does not split.} \tag{GEOM2}$$

Figure 1.4 on page 18 shows a 3-cyclic covering with only one boundary lying above the outer boundary of the annulus but three above the inner one.

4.3. **Sufficiently Precise Approximation.** As in the disk case, to obtain a suitable description of the model Y_R that enables us to deduce the maximal separating boundary domain, the defining equation (4.2) has to be rewritten using a good enough approximation of f by a pth power h^p. The critical radius (in the current situation better called 'critical thickness') giving the maximal separating boundary domain can then be read off from the modified Newton polygon of $f - h^p$ via its critical line segment.

4.3.1. *Definition.* The concept of sufficiently precise approximations carries over from the disk case to the current situation of coverings of open annuli. Again, the definition is stated in terms of the critical line segment in the modified Newton polygon of the approximation; our previous definitions (Defs. 2.20 and 2.22) are easy to adapt to the current situation.

Definition 4.4. The *modified Newton polygon* of the Laurent series $f = \sum_{i=-\infty}^{\infty} a_i t^i \in A$ is the lower convex hull over the points $P_i := (i, v(a_i))$, for integers $i \in \mathbb{Z} \setminus \{0\}$, and the specially defined point

$$P_0 := \begin{cases} (0, v(a_0)) & \text{if } v(a_0) < p/(p-1)\,, \\ P_0' = (0, p/(p-1)) & \text{otherwise}\,. \end{cases}$$

We call

$$k := \min\{i \in \mathbb{Z} \mid v(P_i) \le v(P_j) \text{ for all } j \in \mathbb{Z}\}$$

the *critical index* of f.[14] The corresponding point P_k is called the *critical point* of f; it is the leftmost of all lowest-valued points. If there are line segments with negative slope, the line segment $\overline{P_l P_k}$ with negative slope of smallest absolute value is called the *critical line segment* (it will necessarily have the critical point as right vertex point).

As in Def. 2.25, we want to call approximations sufficiently precise, if they give rise to a critical line segment basically not involving p-coefficients. Due to the more general situation in this section, we have to state the conditions with a bit more care.

Definition 4.5. Let the Laurent series $f \in A^\times$ be as in the geometric situation (GEOM2). Let $h \in A^\times$ be given. Denote by P_k the critical point in the modified Newton polygon of $f - h^p$ and by $\overline{P_l P_k}$ the critical line segment (provided there is one). The approximation of f via h is called *sufficiently precise* if one of the following conditions is satisfied:

(1) m prime to p and either there is no critical line segment or $\operatorname{slope}(\overline{P_l P_k}) \le -\epsilon$,
(2) m prime to p and $\operatorname{slope}(\overline{P_l P_k}) > -\epsilon$ and
  (a) $P_0' \in \overline{P_l P_k}$ or
  (b) $P_0' \notin \overline{P_l P_k}$ and $l \in \mathbb{Z}$ prime to p,
(3) $m = 0$ and $l \in \mathbb{Z}$ prime to p.

We call (1) the trivial case, (2a) and (3) the non-split resp. split separable case, and (2b) the inseparable case; cf. Fig. 4.2.

[14] Since $f \in A$ satisfies the growth condition (4.1), a well-defined minimum always exists.

Again, the existence of sufficiently precise approximations is the heart of our proof of Thm. 4.2. The assertion, that sufficiently precise approximations exist, is the contents of the next proposition; it will take some work to establish this result. In Sect. 4.5.1, we adapt the method of p-Taylor expansion to the current situation, and this will in most cases enable us to produce sufficiently precise approximations of f within finitely many steps. There are some cases, however, that require additional means. For those, we have to adopt the approximation algorithm inherent in the proofs of Prop. 2.2 and Lem. 4.9; see Sect. 4.5.2. The greater complexity of the present situation is due to the fact that the open annulus X has two boundaries and we have to deal with Laurent series instead of power series.

Proposition 4.6. *Let the Laurent series $f \in A^\times$ be as in the geometric situation* (GEOM2)*. Replacing K by a finite field extension, there is a sufficiently precise approximation $h \in A^\times$ of f by a pth power. Both the extension field and the approximation can be determined by a practical algorithm.*

4.3.2. *Critical Thickness.* The analog of Prop. 2.28 holds true in the current situation: essentially, the critical line segment in the modified Newton polygon of $f - h^p$ does not depend on which sufficiently precise approximation h has been chosen; we can therefore use it for the definition of the critical thickness, which we assert to give the maximal separating boundary domain (Prop. 4.12).

Proposition 4.7. *Let the Laurent series $f \in A^\times$ be as in the geometric situation* (GEOM2) *and let $h \in A^\times$ be any sufficiently precise approximation of f by a pth power. Denote by P_k the critical point in the modified Newton polygon of $f - h^p$ and by $\overline{P_l P_k}$ the critical line segment (provided there is one). The case of Def. 4.5, into which the critical point resp. the critical segment falls, does not depend on which sufficiently precise approximation has been chosen and neither does m. In the split separable case (3) and in the inseparable case (2b), also l does not depend on the approximation chosen.*

Definition 4.8. Let the Laurent series $f \in A^\times$ be as in the geometric situation (GEOM2). Denote by P_k the critical point in the modified Newton polygon of some sufficiently precise approximation of f by a pth power and by $\overline{P_l P_k}$ the critical line segment (provided there is one). The *critical thickness* of f (with respect to η_1) is

$$\rho_0 := \begin{cases} \epsilon & \text{if there is no critical segment}\,, \\ \min\{|\mathrm{slope}(\overline{P_l P_k})|, \epsilon\} & \text{otherwise}\,. \end{cases}$$

In other words, the critical thickness essentially is the absolute value of the slope of the critical line segment (as long as meaningful).

Proposition 4.7 will be an easy consequence of Lem. 4.9, a technical lemma that will be used several times throughout this section and is, in a way, a sophisticated adaption of the uniqueness result from Prop. 2.2 (which was used to prove the assertion in the case of disk covers, see Prop. 2.28 and its proof).

We will use analog notation as in Sect. 2.3.1 and consider the discrete valuation v_ρ corresponding to a radius[15] $\rho \in v(K^\times)$, with $0 < \rho \leq \epsilon$, and satisfying

$$v_\rho(t) = \rho\,.$$

[15] This is with respect to the boundary under consideration.

We will also consider different extensions to rank-two-valuations, depending on whether v_ρ is seen as a valuation corresponding to the inner boundary of the boundary domain

$$\mathsf{A}_\rho := \{x \in \mathsf{X} \mid v(t(x)) < \rho\}\ ,$$

or as a valuation on the affinoid

$$\mathsf{B}_\rho := \{x \in \mathsf{X} \mid \rho = v(t(x))\}$$

(for $\rho < \epsilon$). In the former case, we have the rank-two-valuation v_ρ^- that is associated to the inner boundary of A_ρ in the sense of Sect. 1.3.5; in the latter case, we consider the rank-two-valuation v_ρ^+, whose second component corresponds to the place $\bar{t} = 0$ of the residue field $\operatorname{Frac} k[\bar{t}]$.

Lemma 4.9. *Let $f \in A^\times$ be as in the geometric situation (GEOM2) and let $\eta \in \partial X$ be a boundary point over which there is only one boundary of Y. Then:*

(1) *The set*

$$\{v_\eta(f - h^p) \mid h \in A\} \subset \mathbb{Q} \times \mathbb{Z}$$

has a unique maximum (μ, m), with $\mu \le p/(p-1)$ and with m prime to p. Moreover, for any $\rho \in v(K^\times)$ with $0 < \rho \le \epsilon$, the set

$$\{v_\rho(f - h^p) \mid h \in A \text{ with } v_\eta(f - h^p) = (\mu, m)\} \subset v(K^\times) \tag{4.4}$$

also takes a unique maximum $\tilde{\mu}$, with $\tilde{\mu} \le \mu + m\rho$.

(2) *If $\mu + m\rho \le p/(p-1)$ holds true and if the extension of v_ρ to $\operatorname{Frac} B$ is weakly unramified, the Kummer equation (4.2) has irreducible reduction both over the inner boundary of A_ρ and, provided $\rho < \epsilon$, over the affinoid B_ρ.*
   (a) *In the case $\tilde{\mu} = p/(p-1)$, the reduction is given by an Artin–Schreier equation and the set $\{v_\rho^-(f - h^p) \mid h \in A^\times\}$ has the unique maximum $(p/(p-1), -m)$.*
   (b) *In all other cases, the reduction is given by a purely inseparable equation and the set $\{v_\rho^+(f - h^p) \mid h \in A^\times\}$ has a unique maximum $(\tilde{\mu}, \tilde{m})$, with $\tilde{m} \le m$ and $\tilde{m}$ prime to p.*

Proof. By assumption, the covering given by Eq. (4.2) does not split over $\eta \in \partial\mathsf{X}$; that is, Eq. (4.2) remains irreducible over the residue field of $v_\mathfrak{p}$ (with $v_\mathfrak{p}$ denoting the discrete valuation associated to $\eta \in \partial\mathsf{X}$, as usual). The same arguments as in the proof of Prop. 2.2 then show that $\{v_\mathfrak{p}(f - h^p) \mid h \in A\}$ takes a unique maximum $\mu \le p/(p-1)$. In contrast to the disk case, a best approximation with value exactly $p/(p-1)$ is possible, as we have to deal with Laurent series instead of power series and this can lead to an irreducible Artin–Schreier equation of the form $\bar{y}^p + \bar{y} = \bar{g}$, with $\bar{g} \in k((\bar{t})) \setminus k[\![\bar{t}]\!]$.

Since open analytic curves are defined in terms of permanent models and the discrete valuations corresponding to the boundaries are therefore weakly unramified, reasoning as in the proof of Prop. 2.2 shows that the second component of v_η also attains a maximum $m \in \mathbb{Z}$, when running over all elements $h \in A^\times$ giving the best possible approximation μ with respect to the first component. This is because both in the case of purely inseparable reduction (that is, $\mu < p/(p-1)$) and in the case of separable reduction (that is, $\mu = p/(p-1)$) unwanted p-coefficients can be eliminated, so that the maximal value m that can occur is characterized by the fact that it is prime to p. For each $\rho \in v(K^\times)$, the set (4.4) is bounded from above by $\mu + m\rho$, corresponding to the coefficient with index m. As v_ρ is discrete, a maximal value $\tilde{\mu}$ is thus attained. This shows (1).

For (2), suppose that $\mu + m\rho \le p/(p-1)$ holds true and that the extension of v_ρ to $\operatorname{Frac} B$ is weakly unramified (by Epp's result, this could in theory be realized through replacing K by a sufficiently large finite extension). Let $\tilde{h} \in A^\times$ be a maximal element as in (1), that is, an element with $v_\eta(f - \tilde{h}^p) = (\mu, m)$ and $v_\rho(f - \tilde{h}^p) = \tilde{\mu} \le \mu + m\rho$. Due to our assumption on v_ρ, there is $\lambda \in K$ with valuation $v(\lambda) = \tilde{\mu}/p$. Then the usual variable change $w := (y - \tilde{h})/\lambda$ transforms Eq. (4.2) into

$$w^p + \cdots + p\tilde{h}^{p-1}\lambda^{1-p} w = \cdots + ut^{\tilde{m}} + \text{ terms of positive valuation}\,, \tag{4.5}$$

with $\tilde{m} \le m$ and $v_\rho(ut^{\tilde{m}}) = 0$, and where 'positive valuation' is with respect to v_ρ. Equation (4.5) reduces to an Artin–Schreier equation resp. to a purely inseparable equation, depending on whether $\tilde{\mu}$ is equal to $p/(p-1)$ or is strictly smaller. In any case, the reduction is seen to be irreducible—both over $\operatorname{Frac} k[\![\bar{t}^{-1}]\!]$ (the residue field corresponding to the inner boundary of A_ρ) and over $\operatorname{Frac} k[\bar{t}]$ (the residue field corresponding to the valuation on B_ρ). Otherwise, the approximation by $\tilde{h}$ could be improved as in the proof of Prop. 2.2—contradicting the fact that $\tilde{h}$ is supposed to be a best approximation with respect to v_ρ. The crucial point is that this only involves terms of order smaller than m/p and therefore the best approximating property of $\tilde{h}$ with respect to v_η will not get lost.

In the above way, we can eliminate unwanted p-coefficients. We obtain: in the case of separable reduction (that is, if $\tilde{\mu} = \mu + m\rho = p/(p-1)$), the maximal possible value for $v_\rho^-(f - h^p)$ is $(p/(p-1), -m)$; in the case of inseparable reduction, the maximal possible value for $v_\rho^+(f - h^p)$ is $(\tilde{\mu}, \tilde{m})$, with $\tilde{m} \le m$ prime to p. □

Proof of Prop. 4.7. With the formula of Def. 4.8, define the critical thickness ρ_0 corresponding to (and a priori depending on) the sufficiently precise approximation $h \in A^\times$.

If the covering given by the Kummer equation (4.2) does not split over the outer boundary $\eta_1 \in \partial\mathsf{X}$, the set $\{v_{\eta_1}(f - \tilde{h}^p) \mid \tilde{h} \in A\}$ takes a unique maximum (μ, m), with m prime to p; see Lem. 4.9. It is clear that $P_k = P_m = (m, \mu)$ holds true; in particular, the critical point is then uniquely determined by the covering. Finitely enlarging the base field K (if necessary), we can assume by Epp's result [Epp73] that the extension of v_{ρ_0} to $\operatorname{Frac} B$ is weakly unramified. We are hence in the situation of Lem. 4.9, which will show that all relevant information concerning the critical point resp. the critical segment is determined by the covering (that is, by f) and does not depend on the chosen sufficiently precise approximation.

More precisely, by the lemma, there is a unique maximum

$$\tilde{\mu} := \max\{v_{\rho_0}(f - \tilde{h}^p) \mid \tilde{h} \in A \text{ with } v_{\eta_1}(f - \tilde{h}^p) = (\mu, m)\} \le \mu + m\rho_0\,;$$

by definition of ρ_0, we always have $\mu + m\rho_0 \le p/(p-1)$ and $\tilde{\mu} = \mu + m\rho_0 = v_{\rho_0}(f - h^p)$. According to Def. 4.5, we distinguish several cases. When the approximation by h falls into case (2b), we have $\tilde{\mu} = \mu + m\rho_0 < p/(p-1)$. Then $v_{\rho_0}^+(f - \tilde{h}^p)$ attains a unique maximum $(\tilde{\mu}, \tilde{m})$, when running over all elements $\tilde{h}$ giving a best approximation with respect to v_{η_1}; we have $\tilde{m} \le m$ prime to p. As $\tilde{m}$ is characterized by this fact and as $v_{\rho_0}^+(f - h^p) = (\tilde{\mu}, l)$, the maximal possible value $\tilde{m}$ has to coincide with l. As a consequence, P_l is then also uniquely determined by the covering and so is then, in particular, the critical thickness ρ_0.

The other cases are similar. In case (2a), we have $\mu + m\rho_0 = p/(p-1)$ and we are led to a separable Artin–Schreier equation in reduction; though l is no longer uniquely determined, the slope corresponding to the critical line segment is still determined just in terms of the covering, as we have $\mathrm{slope}(\overline{P_l P_m}) = \mathrm{slope}(\overline{P_0' P_m})$. In case (1), application of the lemma with respect to $\rho_0 = \epsilon$ shows that the best approximation of f by a pth power with respect to v_ϵ is of value $\mu + m\epsilon$ and that $(\mu + m\epsilon, -m)$ is the maximal value for v_ϵ^-; consequently, all sufficiently precise approximations will either give no critical segment at all or a critical line segment of slope smaller or equal to $-\epsilon$.

Finally, when the covering splits over the outer boundary $\eta_1 \in \partial\mathsf{X}$, we necessarily have $P_k = P_0'$ and we are in case (3). By assumption, the covering cannot also split over the inner boundary $\eta_2 \in \partial\mathsf{X}$. To show that l—and then also the slope of the critical line segment—are uniquely determined, we can hence argue as in (2a): Lem. 4.9 has to be applied with respect to η_2 and the correctly interpreted critical thickness $\tilde{\rho}_0 = \epsilon - |\mathrm{slope}(\overline{P_l P_0'})|$. □

Remark 4.10. The above proof shows that, in the case where the covering does not split over the outer boundary, the critical point of any sufficiently precise approximation corresponds to the value (μ, m) of a best approximation in the sense of Lem. 4.9. We will therefore usually denote the critical point resp. the critical segment (provided there is one) of sufficiently precise approximations by P_m resp. $\overline{P_l P_m}$.

The next corollary is an immediate consequence of Prop. 4.7 and its proof, and it is an important ingredient for our proof of Prop. 4.12, which explicitly describes the maximal separating boundary domain for the covering $\phi : \mathsf{Y} \to \mathsf{X}$; namely, the corollary is the key result for showing that A_{ρ_0} is maximal separating, as it gives an explicit description of the reduction over the inner boundary of A_{ρ_0} and, if $\rho_0 < \epsilon$, over the affinoid B_{ρ_0}.

Corollary 4.11. *Let $f \in A^\times$ be as in the geometric situation (GEOM2). Denote by P_m the critical point of some sufficiently precise approximation of f and, provided there is one, by $\overline{P_l P_m}$ the critical line segment; the critical thickness shall be denoted ρ_0. We distinguish cases according to Def. 4.5.*

(1) *In the trivial case, the Kummer equation (4.2) has irreducible reduction over the inner boundary of X, given either by $\bar{w}^p = \cdots + \bar{t}^m$ or $\bar{w}^p + \bar{w} = \cdots + \bar{t}^m$, with m prime to p.*

(2) *In the non-split separable case resp. in the inseparable case, Eq. (4.2) has irreducible reduction over the inner boundary of A_{ρ_0} and also over the affinoid B_{ρ_0}. The reduction is given by*
   (a) *an Artin–Schreier equation $\bar{w}^p + \bar{w} = \cdots + \bar{t}^m$ resp. by*
   (b) *a purely inseparable equation $\bar{w}^p = \bar{u}_l \bar{t}^l + \cdots + \bar{t}^m$,*

   with $l < m$ and l, m prime to p (and with $\bar{u}_l$ denoting a unit of k).

(3) *In the split separable case, Eq. (4.2) splits over the inner boundary of A_{ρ_0} but remains irreducible over the affinoid B_{ρ_0}, as the reduction is given by an Artin–Schreier equation $\bar{w}^p + \bar{w} = \bar{t}^l + \cdots$, with $l < 0$ prime to p and the right hand side a polynomial in $\bar{t}^{-1}$ of degree $|l|$.*

(It might be necessary to replace K by a specific finite field extension to be able to write down the respective equations.) □

4.4. **Maximal Separating Boundary Domain.** We can now state what the maximal separating boundary domain will be.

Proposition 4.12. *Let the situation be as in Sect. 4.2; that is, $f \in A^\times$ shall be as in (GEOM2) and define a p-cyclic covering $\phi : Y \to X$ of the open annulus of thickness ϵ. After replacing K by a suitable finite field extension, Prop. 4.6 guarantees the existence of a sufficiently precise approximation $h \in A^\times$ of f. Denote by P_m the corresponding critical point and, provided there is one, by $\overline{P_l P_m}$ the critical segment; the critical thickness defined as in Def. 4.8 shall be denoted ρ_0. Then the boundary domain*

$$\mathsf{A} := \{x \in X \mid v(t(x)) < \rho_0\}$$

is the maximal separating boundary domain with respect to $\eta_1 \in \partial X$.

Proof. We distinguish two cases. When the covering splits over the outer boundary $\eta_1 \in \partial\mathsf{X}$, there are p open annuli lying above any separating boundary domain with respect to η_1. The critical segment then falls into case (3) of Def. 4.5; that is, we have $P_m = P_0'$ and $l < 0$ prime to p. For any $\rho \leq \rho_0$, there are p boundaries lying above the inner boundary[16] of

$$\mathsf{A}_\rho = \{x \in \mathsf{X} \mid v(t(x)) < \rho\} ,$$

and ρ_0 is largest with this property. Indeed, for $\rho < \rho_0$, we have $v_\rho(f - h^p) > p/(p-1)$, so f is a pth power with respect to that boundary; for $\rho = \rho_0$, we get a splitting Artin–Schreier equation over that boundary, see Cor. 4.11. On the other hand, for all $\rho > \rho_0$, the point P_l from the critical line segment prevents the existence of approximations with value $p/(p-1)$ or better (with respect to the inner boundary), so there is only one boundary lying above. All in all, A is recognized to be maximal separating.

When the covering does not split over $\eta_1 \in \partial\mathsf{X}$, any separating boundary domain with respect to η_1 will have a single open annulus lying above. As in the proof of Lem. 2.34, we will use Lem. 1.31 to show that A is separating, or in other words, that $\mathsf{V} := \phi^{-1}(\mathsf{A})$ is an open annulus. Denote by $A_1 := \mathring{\mathcal{O}}_{\mathsf{A}}$ and $B_1 := \mathring{\mathcal{O}}_{\mathsf{V}}$ the zero-bounded analytic functions on A and V, respectively. Recall that B_1 is the integral closure of A_1 in the extension of $\operatorname{Frac} A_1$ given by the Kummer equation (4.2) (considered as an equation over A_1).

We first have to see that there is also only one boundary point of V lying above the inner boundary of A, but this is an immediate consequence of Cor. 4.11—no matter whether the line segment falls into case (1), (2a), or (2b) of Def. 4.5. Let $\lambda \in K$ be some arbitrary element of valuation $v(\lambda) = v_{\mathfrak{p}_1}(f - h^p)/p$ (with $v_{\mathfrak{p}_1}$ denoting the discrete valuation corresponding to the boundary η_1). Then the element $w := (y - h)/\lambda \in B_1$ satisfies the irreducible equation

$$w^p + \cdots + p h^{p-1} \lambda^{1-p} w = \frac{f - h^p}{\lambda^p}$$

over A_1, and therefore

$$\operatorname{Norm}(w) = \frac{f - h^p}{\lambda^p} =: g .$$

With the same reasoning as in the disk case (cf. the proof of Lem. 2.34), it follows from the definition of the critical thickness that $g \in A_1$ can be written as $t^m u$, with $u \in A_1^\times$ a unit and m the index of the critical point; note that m is prime to p. Therefore, w is seen to meet requirements (1) and (2) of Lem. 1.31; we conclude that A is separating.

[16] That is, the boundary distinct from the one induced by $\eta_1 \in \partial\mathsf{X}$.

When $\rho_0 = \epsilon$ holds true, A is obviously maximal with that property. Now assume $\rho_0 < \epsilon$. Then the critical line segment falls into case (2a) or (2b) of Def. 4.5. Corollary 4.11 shows that the affinoid $\phi^{-1}(\mathsf{B})$, the preimage of

$$\mathsf{B} = \{x \in \mathsf{X} \mid \rho_0 = v(t(x))\}, \tag{4.6}$$

has irreducible reduction either described by an Artin–Schreier equation or by a purely inseparable equation of the form

$$\bar{w}^p = \bar{u}_l \bar{t}^l + \cdots + \bar{t}^m \tag{4.7}$$

(with a non-zero element $\bar{u}_l \in k$, with $l < m$, and with l, m prime to p). In the former case, the reduction is smooth; in the latter case, the reduction contains at least one singular point as the differential of the right hand side of Eq. (4.7) has at least one zero distinct from 0 and ∞. As a consequence, Lem. 1.33 shows that A is maximal. □

Up to showing that sufficiently precise approximations always exist and can be constructed by a practical algorithm—both will be shown in the next section—we have explicitly determined the maximal separating boundary domain for the covering ϕ and have thereby also proven Thm. 4.2. □

4.5. **Existence of Sufficiently Precise Approximations.** We still need to see that sufficiently precise approximations of f exist and, moreover, can be obtained by a practical procedure.

4.5.1. *Generalization of Formal p-Taylor Expansion.* The main means to practically find sufficiently precise approximations is again the algorithm of formal p-Taylor expansion (adapted to the current situation of Laurent series). As before, one important aspect of this method is the fact that no prior assumptions on the base field have to be made. In contrast to the disk case, determining the maximal separating boundary domain does not involve finding a suitable center point (as all boundary domains with respect to $\eta_1 \in \partial\mathsf{X}$ can be written in terms of the chosen parameter t). With respect to this issue, we are in a more simple situation and it suffices to consider the non-generic version of the algorithm.

Definition 4.13. Let the situation be as presented in Sect. 4.2; that is, A shall be the ring of zero-bounded analytic functions on the open annulus X of thickness ϵ, and $t \in A$ shall be a fixed parameter with respect to the chosen boundary $\eta_1 \in \partial\mathsf{X}$. A *formal p-Taylor expansion of level* $n \in \mathbb{N}_0$ for $f \in A^\times$ is given by a function $h \in A^\times$ such that the following condition on the coefficients in the Laurent series expansion

$$f - h^p =: \sum_{i=-\infty}^{\infty} a_i' t^i$$

holds true:

(∗) $v(a'_{jp}) \geq v_n + \max\{-jp\epsilon, 0\}$ for all $j \in \mathbb{Z}$.

This is abbreviated to $(f; h, t; a_i')$.

With a formal p-Taylor expansion $(f; h, t; a_i')$, we obtain an approximation of f by a pth power in which all p-coefficients of $f - h^p$ are simultanously small. More precisely, when the level is $n \in \mathbb{N}_0$, the valuation of all p-coefficients will be at least v_n above 'ground level' (meaning the minimal valuation the respective coefficient needs to have by (4.1) to give an element of A). In other words, all p-coefficients can be raised within a tubelike area of diameter $p/(p-1)$; Figure 4.2 shows this tubular neighborhood

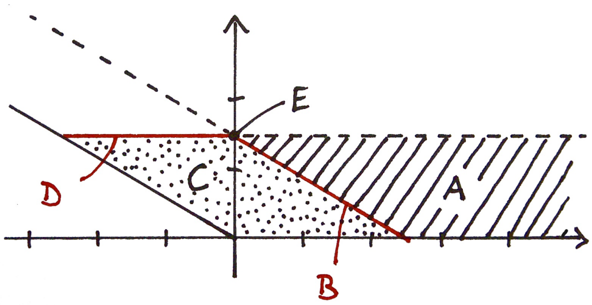

FIGURE 4.2. When the covering ϕ : Y → X does not split over $\eta_1 \in \partial$X, the critical point P_m from a sufficiently precise approximation of f lies in one of the areas marked (A), (B), (C), or (D); otherwise, $P_m = P_0'$ is the point (E). Depending on this, only specific cases of Def. 4.5 can occur: for $P_m \in$ (A), only cases (2a) and (2b) are possible; for $P_m \in$ (B) ∪ (C) ∪ (D), the only possible cases are (1) and (2b); the point (E) corresponds to case (3). Ultimately, this also determines in which way a sufficiently precise approximation can be obtained; see the proof of Prop. 4.6 in Sect. 4.5.3.

(partitioned into several subareas, which play a role when proving the existence of sufficiently precise approximations in Sect. 4.5.3).

Proposition 4.14. *With notation as in Def. 4.13, let $f \in A^\times$ be given and fix a level $n \in \mathbb{N}_0$. After replacing K by a suitable finite field extension, there exists a formal p-Taylor expansion $(f;h,t;a_i')$. Both the extension field and the expansion can be determined by a practical algorithm.*

Remark 4.15. With notation as in Lem. 2.10, the proof of Prop. 4.14 will show that an extension field as in the statement of the proposition is given by

$$L := (\dots(K_0)_1)\dots)_n = K(\Pi \mid \Pi^{pN} = \pi)\,,$$

with π a uniformizer of R and with $N = 1 + \dots + (n+1) = (n+1)(n+2)/2$.

Remark 4.16. In contrast to the disk case, we cannot assume p-Taylor expansions to satisfy $a_0' = 0$. The reason is that we have to deal with Laurent series instead of power series and there might be mixed terms interfering with the zeroth coefficient, see the proof of Prop. 4.14. This can lead to an additional twist when constructing sufficiently precise approximating functions; see the proof of Prop. 4.6 in Sect. 4.5.3.

Proof of Prop. 4.14. The proof is completely analog to the proof of Prop. 2.12: we inductively construct elements $h^{[n]}$, whose pth powers approximate f up to level $n \in \mathbb{N}_0$. As before, it will be necessary in each step to pass to a successively larger finite extension $K^{[n]}$ of K, with corresponding ring of integers $R^{[n]}$; the zero-bounded analytic functions on X considered as a $K^{[n]}$-curve will be denoted $A^{[n]}$. For notational convenience, we also set $K^{[-1]} := K$ and $R^{[-1]} := R$, as well as $A^{[-1]} := A$.

For $n \in \mathbb{N}_0$, we then define

$$h^{[-1]}(t) := 0 \in A^{[-1]} \quad \text{and} \quad h^{[n]}(t) := h^{[n-1]}(t) + \delta^{[n]}(t) \in A^{[n]}\,, \tag{4.8}$$

with the improvement term

$$\delta^{[n]}(t) := \sum_{j=-\infty}^{\infty} b_j^{[n]} t^j \in A^{[n]} \tag{4.9}$$

having coefficients $b_j^{[n]} \in R^{[n]}$, the pth powers of which approximate the p-coefficients $a_{jp}^{[n-1]}$ from the previous approximation step

$$f(t) - (h^{[n-1]}(t))^p =: \sum_{i=-\infty}^{\infty} a_i^{[n-1]} t^i \in A^{[n-1]} \tag{4.10}$$

at least up to valuation $v_n + \max\{-jp\epsilon, 0\}$, that is,

$$v(a_{jp}^{[n-1]} - (b_j^{[n]})^p) \geq v_n + \max\{-jp\epsilon, 0\} . \tag{4.11}$$

Note that since $f(t)$ satisfies the growth condition (4.1), the constructed elements $h^{[n]}(t)$ also do and are thus indeed elements of $A^{[n]}$. Moreover, $h^{[n]}(t)$ is a unit of $A^{[n]}$ as $a_0^{[n]} \in R^{[n]}$ is seen to be of valuation zero.

By Lem. 2.10, all elements $b_j^{[n]}$ as required are contained in the ring of integers $R^{[n]}$ of the finite extension

$$K^{[n]} := (K^{[n-1]})_n$$

of degree p^{n+1} (with notation as in the lemma). Unlike in the disk case, inequality (4.11) requires the full strength of Lem. 2.10. Of course, other field extensions can work as well; in particular, when f is polynomial, we could also brutally adjoin all pth roots

$$b_j^{[n]} := (a_{jp}^{[n-1]})^{1/p}$$

needed and use these for the definition of $\delta^{[n]}(t)$.

Precisely as in the proof of Prop. 2.12, one shows that $h^{[n]}$ provides an approximation of level n by calculating

$$\begin{aligned} f - (h^{[n]})^p &= f - (h^{[n-1]} + \delta^{[n]})^p \\ &= (f - (h^{[n-1]})^p) - (\delta^{[n]})^p - \sum_{j=1}^{p-1} \binom{p}{j} (h^{[n-1]})^j (\delta^{[n]})^{p-j} \end{aligned} \tag{4.12}$$

and keeping track of the valuation of the involved terms (again using the fact that the involved functions satisfy the growth condition (4.1)).

Observe that, in contrast to the disk case, we can in general not achieve that $h^{[n]}(t)^p$ kills the zeroth coefficient of $f(t)$. This is because multiplication of terms with negative and positive powers of t might very well result in expressions also contributing to the zeroth coefficient. The estimate $v(a_0') \geq v_n$ from property $(*)$ of Def. 4.13 is the best we can get with this algorithm. □

Though $v_n \to p/(p-1)$ for $n \to \infty$, we always have $v_n < p/(p-1)$ for finite level $n \in \mathbb{N}_0$. Note that in the current situation, best approximations might very well have value $p/(p-1)$; see Lem. 4.9. The consequence is that the p-Taylor algorithm will not always be enough to produce sufficiently precise approximations, as the critical segment might keep involving unwanted p-coefficients—no matter how high the level is chosen; cf. Fig. 4.3. To be able to prove that sufficiently precise approximations can always be obtained, we first have to establish some facts concerning the approximation algorithm from the proof of Lem. 4.9.

4.5.2. *Potentially Separable Reduction.* As remarked above, the p-Taylor algorithm will not always be enough to produce sufficiently precise approximations of f. Namely, this happens whenever the covering has potentially separable reduction over one of the boundaries involved and we have to deal with approximations of valuation $p/(p-1)$ or larger. In this situation, the naive approximation algorithm inherent in the proofs of Prop. 2.2 and Lem. 4.9 will prove very useful, as it can also handle approximations with value $p/(p-1)$ or larger. We recall how the algorithm works.

Let the situation be as before; that is, the Laurent series $f \in A^\times$ shall satisfy (GEOM2) and define via Eq. (4.2) the p-cyclic covering $\phi : \mathsf{Y} \to \mathsf{X}$ of the open annulus X of thickness ϵ. When the covering does not split over $\eta_1 \in \partial\mathsf{X}$, we want to approximate f by a pth power as good as possible—that is, maximize $v_{\eta_1}(f - h^p)$, which we know in that situation to be not larger than $p/(p-1)$ (see Lem. 4.9); when the covering splits over η_1, we want to find an approximation with value better than $p/(p-1)$.

Algorithm 4.17. With setting and notation as above, assume that K is large enough to have the extensions of v_{η_1} to $K(\mathsf{Y})$ be weakly unramified. We start with some arbitrary $h \in A^\times$ and set $\mu := v_{\eta_1}(f - h^p)$. Suppose that $\mu < p/(p-1)$. Due to assumption on K, there is $\lambda \in K$ with valuation $v(\lambda) = \mu/p$. Then the variable change $w := (y - h)/\lambda$ leads to an equation reducing to a purely inseparable equation of the form $\bar{w}^p = \bar{g}$ over $\operatorname{Frac} k[\![\bar{t}]\!]$ (here we have written $g := (f - h^p)/\lambda^p$). Precisely if $\bar{g}$ is a pth power—say, $\bar{g} = \bar{q}^p$—the approximation can be improved through replacing h by $h + \lambda q$, giving an approximation with value strictly larger than μ. The analog reasoning applies when $\mu = p/(p-1)$. In that case, the above variable change leads to an equation reducing to an Artin–Schreier type equation $\bar{w}^p + \bar{w} = \bar{g}$. Precisely if there is $\bar{q}$ with $\bar{q}^p + \bar{q} = \bar{g}$, the initial approximation can be improved through replacing h by $h + \lambda q$, and this gives an approximation with value strictly larger than $p/(p-1)$. As the base field K is discretely valued, finitely many of these improvement steps suffice to either produce a best possible approximation of f by a pth power with value $\mu \le p/(p-1)$ or some approximation with value strictly larger than $p/(p-1)$.

For the above to work, it is crucial that—right from the start—the base field is large enough to contain elements λ with all required valuations; if one had to enlarge K during this algorithmic procedure, the value group of the base field would get larger, and all work would be in vain. In the theoretical setting of a finite covering $\mathsf{Y} \to \mathsf{X}$ of open analytic curves (that is, in the setting Prop. 2.2 and Lem. 4.9 were proven), this poses no problem as open analytic curves come by definition from permanent models and the involved valuations are weakly unramified. But for practical applications, the algorithm would only be useful if we could state a sufficiently large extension field in advance. As remarked before, this is in general not possible and one of the reasons why we rely on formal p-Taylor approximations. Fortunately, precisely when the latter is not enough to produce sufficiently precise approximations, Alg. 4.17 will turn out to be of practical value because it will then be possible to explicitly state a finite field extension K'/K over which the extensions of v_{η_1} to $K(\mathsf{Y})$ will be weakly unramified.

For the argumentation to follow, we temporarily abandon our usual notation from Sect. 4 and assume to come from the global situation of a p-cyclic covering of the projective line, $\Phi : Y \to X = \mathbb{P}^1_K$, given by a Kummer equation $y^p = f$, with $f \in R[t]$. We will show that K' has to be chosen so large that Y contains enough K'-rational points specializing to each component in the reduction of Y, and with these specializations not coinciding with the reduction of ramification points. To make this more precise, we choose an affinoid subdomain $\mathsf{U} \subset X^{\mathrm{rig}}$ with good reduction and which does not contain any ramification points of Φ. For example, we could define U as the complement in X^{rig} of all residue classes containing ramification points. We may assume U to be connected. Then the reduction $\bar{U}$ of U is a smooth irreducible affine k-curve. Let $\mathsf{V} := \Phi^{\mathrm{rig}^{-1}}(\mathsf{U}) \subset Y^{\mathrm{rig}}$ denote the preimage of U. In general, V does neither have good nor potentially good reduction; however, when V has potentially separable reduction—which is what we assume from now on—V does not only have potentially good reduction, but it is also easy to determine a finite extension K'/K over which this good reduction is attained.

Lemma 4.18. *Let the situation be as described. Choose some K-rational point $x \in U$.*[17] *Then the residue field K' of some arbitrary point in the Φ^{rig}-preimage of x is a finite extension of K over which V has separable reduction. In particular, Alg. 4.17 can be used over K' to produce within finitely many steps a best approximation of f by a pth power with respect to the Gauß valuation on U.*

Proof. By definition of U, the residue class X in which x lies does not contain ramification points of Φ, so $f(x)$ is a unit of R. Changing parameters, we may assume $x = 0$. Then the constant coefficient $a_0 \in R$ of $f = \sum_{i=0}^{<\infty} a_i t^i$ is a unit, and over the finite extension $K' := K(\alpha \mid \alpha^p = a_0)$—that is, the residue field of points lying above x—we can normalize f such that $a_0 = 1$.

Restricting Φ^{rig} to the preimage $\mathsf{Y} := \Phi^{\mathrm{rig}^{-1}}(\mathsf{X})$, we are in the well-known situation of an étale covering $\phi := \Phi^{\mathrm{rig}}|_{\mathsf{Y}} : \mathsf{Y} \to \mathsf{X}$ of the open unit disk; the covering is given by $y^p = f$, with f considered as a power series over the integral closure R' of R in K'. The results of Sect. 2 imply that f can either be best approximated by a pth power with value strictly smaller than $p/(p-1)$, or ϕ decomposes after passing to a further finite extension K''/K' over which Y has a permanent model. As the former situation is excluded by our assumptions on V, the latter is the case—that is, f is a pth power in $R''[\![t]\!]$ (with R'' denoting the integral closure of R in K'').

We claim that f is then already a pth power in $R'[\![t]\!]$, so the extension K''/K' was not necessary. To see this, we may assume K''/K' to be Galois. Let $\sigma \in \mathrm{Gal}(K''/K')$ be any of the Galois automorphisms and denote by $g := \sum_{i=0}^{\infty} b_i t^i \in R''[\![t]\!]$ a pth root of f. Then

$$\sum_{i=0}^{\infty} \sigma(b_i) t^i = \sigma(g) = \zeta_p g$$

holds true, with $\zeta_p \in K$ a pth root of unity; that is,

$$\sigma(b_i) = \zeta_p b_i\,, \quad \text{for all } i \in \mathbb{N}_0\,. \tag{4.13}$$

As $b_0^p = 1$ holds true, b_0 is a pth root of unity and as such contained in K. It then follows from considering Eq. (4.13) for $i = 0$ that $\zeta_p = 1$. Since $\sigma \in \mathrm{Gal}(K''/K')$ was arbitrary, this implies that $g \in R'[\![t]\!]$, as asserted.

Our arguments show that over the finite extension K'/K, the preimage Y of X decomposes into p distinct open disks, which are the residue classes of p distinct points

[17] Since the affinoid U has good reduction and since the residue field of K is algebraically closed, U always contains K-rational points.

on the reduction of V. Consequently, the reduction of the covering $\mathsf{V} \to \mathsf{U}$ is generically étale over K'. □

The above reasoning requires to have some affinoid U with good reduction. Whenever we come from the global situation—for example, in the course of applying our resolution algorithm—this is the case and poses no problem. In particular, we can then assume the valuation at the boundary of an open annulus to have weakly unramified extensions—provided there is potentially separable reduction over that boundary; if this is the case, Alg. 4.17 can be used to produce within finitely many steps approximations with value $p/(p-1)$ or greater.

But we are also fine in situations arising by subsequent local studies. For example, we have seen in Sect. 4.1 that we can reduce the study of a ramified disk covering to the study of an étale covering of some open annulus A by considering the affinoid disk $\mathsf{D}_{\mathrm{ram}}$ containing all ramification points of the covering. In this situation, we can define $\mathsf{U} \subset \mathsf{D}_{\mathrm{ram}}$ to be the complement of the residue classes of ramification points and choose K so large that the preimage of U contains a K-rational point; in practice, when the covering is given by an equation of the form $y^p = f$, this means to adjoin a pth root of $f(x)$ to K, for some $x \in \mathsf{U}$. The base field will then be large enough to have the covering attain separable reduction over the inner boundary of A—provided it has potentially separable reduction there. Or, in the situation Sect. 4 is mainly devoted to (that is, the study of a p-cyclic covering $\phi : \mathsf{Y} \to \mathsf{X}$ of the open annulus X of thickness ϵ), we can explicitly determine a finite extension K'/K such that Alg. 4.17 can be successfully applied with respect to the valuation v_ρ for some fixed $\rho \in v(K^\times)$, with $0 < \rho < \epsilon$—provided the covering has potentially separable reduction over the affinoid $\mathsf{B}_\rho = \{x \in \mathsf{X} \mid \rho = v(t(x))\}$.

The results that we need for our proof of Prop. 4.6 are summerized in the following corollary.

Corollary 4.19. *Let the situation be as in Sect. 4.2; that is, $\phi : Y \to X$ shall be a p-cyclic étale covering of the open annulus X of thickness ϵ.*

(1) *Assume ϕ to come from the global situation in the course of applying our resolution algorithm (or else, assume to start with permanent models). We can then determine a finite extension K'/K such that the following holds true: if ϕ has potentially separable reduction over the boundary $\eta \in \partial X$, the separable reduction is attained over K'.*

(2) *For fixed radius $\rho \in v(K^\times)$, with $0 < \rho < \epsilon$, we can determine a finite extension K'/K such that the following holds true: if ϕ has potentially separable reduction over the inner boundary of A_ρ, the separable reduction is attained over K'.*

Consequently, in these specific situations, Alg. 4.17 proves to be practical (and can be used to produce the desired best approximations). □

4.5.3. *Combining the Algorithms.* We revert to our main setting with notation as in Sect. 4.2; that is, we study the rigid analytic p-cyclic étale covering $\phi : \mathsf{Y} \to \mathsf{X}$ of the open annulus X, with the covering given by $f \in A^\times$ satisfying (GEOM2).

Combining the methods from Sects. 4.5.1 and 4.5.2, we can finally establish the existence of sufficiently precise approximations for f.

Proof of Prop. 4.6. We first suppose that the covering given by f does not have potentially separable reduction over the outer boundary $\eta_1 \in \partial\mathsf{X}$. In particular, the covering does then not split over η_1. By Lem. 4.9, $\{v_{\eta_1}(f - h^p) \mid h \in A\}$ takes a unique maximum

(μ, m), with $\mu < p/(p-1)$ and m prime to p. So if sufficiently precise approximations of f are to exist, $P_m = (m, \mu)$ necessarily has to be the corresponding critical point. As in the proof of Cor. 2.16, any formal p-Taylor expansion $(f; h, t; a_i')$ of level $n \in \mathbb{N}_0$ high enough to have $v_n > \mu$ will satisfy

$$(\mu, m) = v_{\eta_1}(f - h^p)$$

and extract the correct critical point P_m. Note that P_m falls into one of the areas marked (A), (B), or (C) in Fig. 4.2; we distinguish cases accordingly.

When $P_m \in (\mathrm{A}) \cup (\mathrm{B})$,

$$\tilde{\rho}_0 := |\mathrm{slope}(\overline{P_0' P_m})| = \frac{p/(p-1) - \mu}{m} \leq \epsilon$$

corresponds to the steepest slope the critical line segment in a sufficiently precise approximation of f could have. This is because we always have $v(P_0) \leq p/(p-1)$ by definition of the modified Newton polygon (Def. 4.4). When the covering does not have potentially separable reduction over the inner boundary of the boundary domain

$$\mathsf{A}_{\tilde{\rho}_0} := \{x \in \mathsf{X} \mid v(t(x)) < \tilde{\rho}_0\},$$

Lem. 4.9 shows that there are best approximations $\tilde{h}$ of f with respect to v_{η_1} that in addition satisfy

$$v_{\tilde{\rho}_0}^{+}(f - \tilde{h}^p) = (\tilde{\mu}, \tilde{m}), \quad \text{with } \tilde{\mu} < p/(p-1) = \mu + m\tilde{\rho}_0 \text{ and } \tilde{m} < m \text{ prime to } p.$$

Accordingly, the line through the points $(\tilde{m}, \tilde{\mu})$ and $(m, p/(p-1))$ resp. $(\tilde{m}, \tilde{\mu} - \tilde{m}\tilde{\rho}_0)$ and (m, μ) will intersect the $v(\cdot)$-axis at a value

$$\alpha := \frac{p}{p-1} - m \cdot \frac{p/(p-1) - \tilde{\mu}}{m - \tilde{m}} < p/(p-1).$$

It follows that all p-Taylor expansions $(f; h, t; a_i')$ of level $n \in \mathbb{N}_0$ with $v_n > \alpha$ will give a critical segment $\overline{P_l P_m}$ in the modified Newton polygon of $f - h^p$, with $l < m$ and l prime to p. This is because h cannot provide a better approximation with respect to $v_{\tilde{\rho}_0}$ than $\tilde{h}$ does, and all p-coefficients of $f - h^p$ are by property $(*)$ of p-Taylor expansions (Def. 4.13) too small to be involved in the critical segment. Note that applying the theoretical Lem. 4.9 might have required to work over a sufficiently large finite extension of K; however, this does not make our algorithm less practical: the result that, in the current situation, the p-Taylor algorithm will after finitely many steps yield a sufficiently precise approximation falling into case (2b) of Def. 4.5 is independent of this.

On the other hand, when the covering has potentially separable reduction over the inner boundary of $\mathsf{A}_{\tilde{\rho}_0}$, we can assume by Cor. 4.19 that the base field K is so large that Alg. 4.17 can be applied over K with respect to $v_{\tilde{\rho}_0}$. As a consequence, the reasoning of Lem. 4.9 becomes practical; that is, starting with a best approximation of f with respect to v_{η_1} (which, as above, can be practically obtained by a p-Taylor expansion of level n with $v_n > \mu$), we can improve the approximation within finitely many steps to obtain an element $\tilde{h} \in A$ with

$$v_{\eta_1}(f - \tilde{h}^p) = (\mu, m) \quad \text{and} \quad v_{\tilde{\rho}_0}(f - \tilde{h}^p) = p/(p-1).$$

In other words, the corresponding critical segment $\overline{P_l P_m}$ passes through P_0' and we have obtained a sufficiently precise approximation of f falling into case (1) or (2a) of Def. 4.5. As an aside, the fact that P_0' lies on the critical segment is the reason why in the present situation, formal p-Taylor expansions are generally not enough to

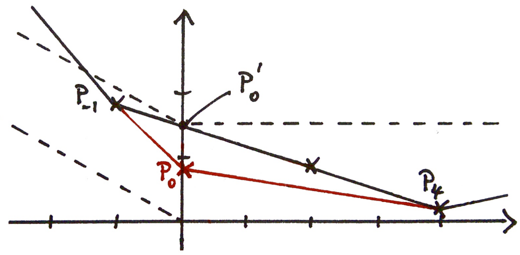

FIGURE 4.3. In those situations in which the critical line segment of a sufficiently precise approximation passes through $P_0' = (0, p/(p-1))$, the p-Taylor algorithm might not be enough to produce a sufficiently precise approximation, as points like $P_0 = (0, v(a_0'))$ could prevent extraction of the correct critical segment.

produce a sufficiently precise approximation of f: the Taylor algorithm cannot force $v(a_0') \geq p/(p-1)$ to have $P_0 = P_0' = (0, p/(p-1))$ hold true; cf. Rem. 4.16 and see Fig. 4.3 for an illustration of the situation.

Next, assume that P_m lies in area (C) of Fig. 4.2. In this case, all p-Taylor expansions $(f; h, t; a_i')$ of level $n \in \mathbb{N}_0$ with

$$v_n > \mu + m\epsilon < \frac{p}{p-1}$$

give rise to sufficiently precise approximations of f, as all p-coefficients will then be too small to interfere with the critical segment: either, there will be no critical segment at all or a critical segment with a slope smaller or equal to $-\epsilon$; or, this segment will be $\overline{P_l P_m}$, with $l < m$ and l prime to p. The former corresponds to case (1) of Def. 4.5, the latter to case (2b).

Now suppose that ϕ has potentially separable reduction over $\eta_1 \in \partial \mathsf{X}$. By Cor. 4.19, we can then assume K to be so large that we can practically apply Alg. 4.17 with respect to v_{η_1}. When ϕ does not split over $\eta_1 \in \partial \mathsf{X}$, this allows to produce within finitely many steps an approximation $\tilde{h} \in A$ satisfying

$$v_{\eta_1}(f - \tilde{h}^p) = (p/(p-1), m) \quad \text{and } m < 0 \text{ prime to } p\,;$$

when ϕ splits over η_1, we can find an approximation with

$$v_{\mathfrak{p}_1}(f - \tilde{h}^p) > p/(p-1)$$

(here, as usual, $v_{\mathfrak{p}_1}$ denotes the discrete valuation associated to η_1). If sufficiently precise approximations of f are to exist in these situations, the corresponding critical point will necessarily be $P_m = (m, p/(p-1))$ in the former case and $P_m = P_0' = (0, p/(p-1))$ in the latter case (that is, P_m falls into area (D) resp. (E) of Fig. 4.2). And indeed, it is not difficult to see that the approximation given by $\tilde{h}$ can be improved such that a sufficiently precise approximation in the sense of Def. 4.5 is attained. Namely, we can adapt the iterative procedure from the p-Taylor algorithm (formulas (4.8)–(4.11) in the proof of Prop. 4.14) to our current situation: instead of starting the inductive definition with 0, we start with $h^{[-1]} := \tilde{h}$ as initial approximation. The crucial point is that the inductive process does not negatively affect the approximation property with respect to v_{η_1}: to see this, set

$$\tilde{\mu} := v_{\mathfrak{p}_1}(f - \tilde{h}^p) \geq p/(p-1)$$

and observe that all terms involved in the improvement functions $\delta^{[n]}(t)$ have valuation at least $\tilde{\mu}/p$, so that all mixed terms in (4.12) will have valuation at least $1+\tilde{\mu}/p \geq \tilde{\mu}$ (the latter inequality holds because $\tilde{\mu}/p \geq 1/(p-1)$). We assert that after a finite number $n \in \mathbb{N}_0$ of these improvement steps, we will end up with a sufficiently precise approximation for f.

When the covering does not split over η_1, it suffices to choose $n \in \mathbb{N}_0$ so high that

$$\nu_n > p/(p-1) + m\epsilon$$

holds true; note that $m < 0$, so the right hand side is strictly smaller than $p/(p-1)$ and n can be chosen as demanded. Then all p-coefficients will be so small that there is either no critical segment at all or it has a slope smaller or equal to $-\epsilon$ (case (1) of Def. 4.5), or there is a critical segment $\overline{P_l P_m}$, with $l < m$ prime to p (corresponding to case (2b)). On the other hand, when the covering splits over η_1, the assumptions on f imply that the covering does not also split over $\eta_2 \in \partial \mathsf{X}$. Then f can be best approximated over that boundary: denote by $(\tilde{\lambda}, \tilde{l})$ the value of a best approximation of f by a pth power with respect to ν_{η_2}. We have $\tilde{\lambda} < p/(p-1)$ and $\tilde{l} < 0$ prime to p (if $\tilde{l}$ were to be positive, the covering would, in contradiction to our assumption, not split over $\eta_1 \in \partial \mathsf{X}$). Analog to the situation at the beginning of this proof, when the number n of improvement steps is chosen so high that $\nu_n > \tilde{\lambda}$ holds true, the point $P_{\tilde{l}} = (\tilde{l}, \tilde{\lambda} - \tilde{l}\epsilon)$ will be a vertex point in the corresponding modified Newton polygon, although $P_{\tilde{l}}$ does not have to be a vertex point of the critical segment (which could start at some index $l > \tilde{l}$). However, choosing $n \in \mathbb{N}_0$ so high that

$$\nu_n > \frac{p}{p-1} - \frac{p/(p-1) - \tilde{\lambda}}{-\tilde{l}}$$

holds true (which is possible since the right hand side is strictly smaller than $p/(p-1)$), all p-coefficients in the so-precise approximation will lie above the line going through $P_{\tilde{l}}$ and P'_0; the result is a critical segment $\overline{P_l P'_0}$ falling into case (3) of Def. 4.5. □

Remark 4.20. Of course, for a given covering $\phi : \mathsf{Y} \to \mathsf{X}$, we do not know in advance whether the covering has potentially separable reduction over one of the boundaries involved. The above nevertheless constitutes a practical algorithm to determine sufficiently precise approximations, as we can run both of our algorithms strictly parallel. After finitely many steps, either the p-Taylor algorithm gives a critical point P_m falling into one of the areas marked (A), (B), or (C) in Fig. 4.2, or else, Alg. 4.17 applied to the outer boundary of X shows that ϕ has separable reduction over that boundary. In the former case, we continue with the Taylor algorithm and let Alg. 4.17 with respect to $\tilde{\rho}_0 = |\mathrm{slope}(\overline{P'_0 P_m})|$ run parallel; in the latter case, we improve the initial approximation with the Taylor algorithm. In any case, after finitely many steps, we will have explicitly determined both a finite field extension and a sufficiently precise approximation of f that can be defined thereover. In particular, Epp's result [Epp73] becomes constructive also in the current situation.

This also finishes our study of prime-cyclic coverings of open annuli and therewith completes our algorithmic proof of the Semistable Reduction Theorem presented in [AW11]. □

4.6. **Example: Non-Equidistant Prime-Cyclic Galois Cover.** We can use our algorithmic method to calculate the semistable reduction of prime-cyclic coverings even when they do not have the equidistant geometry property. This is of practical relevance: In [GRSS10], Greenberg et al. are interested in the $\mathbb{Q}$-rational points and in the Jacobian of the genus-twelve-curve Y that in affine coordinates is given by the equation

$$y^7 = \frac{x^3 - 2x^2 - x + 1}{x^3 - x^2 - 2x + 1} =: q(x) \tag{4.14}$$

over $\mathbb{Q}[x]$. A description of the stable reduction of Y at the prime $p = 7$ would lead to a clearer understanding of the above issues, as Stoll communicated to us. With our algorithm, we are able to determine the desired stable model despite the fact that Y will be seen to not have equidistant geometry.

4.6.1. *From Global to Local.* As in the example from Sect. 3, we will work over a sufficiently large (but a priori not further specified) finite extension K of $\mathbb{Q}_7$, with corresponding ring of integers R and finite residue field k. In particular, we can assume that all occurring models are permanent. A posteriori, we will be able to deduce the minimal extension needed for realizing the stable reduction of Y, using arguments similar to those of Sect. 3.3.

We interpret Y as a covering $\Phi : Y \to X := \mathbb{P}^1_K$ of the projective line, given by the Kummer type equation (4.14). Taking the standard smooth model $X_R := \mathbb{P}^1_R$ as a first model for X, we need to get a description of Y_R, which is defined as the normalization of X_R in $K(Y)$. Since Eq. (4.14) reduces modulo 7 to the irreducible and purely inseparable equation

$$\bar{y}^7 = \bar{q}(\bar{x}) = \frac{(\bar{x}+4)^3}{(\bar{x}+2)^3}\,, \tag{4.15}$$

the model Y_R is indeed described by Eq. (4.14) (considered as an equation over R); cf. the reasoning using Serre's criterion in the proof of Prop. 2.2.

The critical points—corresponding to the zeros and poles of the differential $\mathrm{d}\bar{q}$—are $\bar{x}_1 = 3 \in k$ and $\bar{x}_2 = 5 \in k$ on the special fiber of X_R. Note that these are also the points, where the six ramification points of Φ specialize to: the three zeros of q specialize to $\bar{x}_1$ and the three poles specialize to $\bar{x}_2$. This has the following consequence (for an elementary argument, using the cross-ratio on the projective line, consider [GHvdP88]).

Remark 4.21. The covering $\Phi : Y \to X$ given by Eq. (4.14) does not have the property of equidistant geometry.

The next step in our algorithm is to examine the residue classes of $\bar{x}_1$ and $\bar{x}_2$ (which are open disks) and their preimages, and to study the corresponding finite coverings of open analytic curves. Note that $\bar{x}_1$ and $\bar{x}_2$ are exchanged by the involution

$$\sigma : Y \to Y\,, \quad (x, y) \mapsto (x^{-1}, y^{-1})$$

(which is compatible with the covering map $\Phi : Y \to X$), so we only have to deal with one of the critical points, say $\bar{x}_1$, as the other critical point behaves the same.

4.6.2. *Ramified Covering of the Open Unit Disk.* Let the nominator and denominator of $q(x)$ be denoted by $r(x)$ and $s(x)$, respectively. The substitution $y_1 := y\,s(x)$ leads to the alternative equation $y_1^7 = r(x)s(x)^6$ for the covering Φ. Since we want to pay attention to the critical point $\bar{x}_1 = 3$, we further substitute $x =: 3 + t$ and obtain

$$y_1^7 = f(t)\,, \tag{4.16}$$

with

$$\begin{aligned} f = {} & 33787663 + 363867140t + 1833730444t^2 + 5752789575t^3 \\ & + 12606899422t^4 + 20525955749t^5 + 25780233433t^6 \\ & + 25597791097t^7 + 20426318570t^8 + 13243325186t^9 \\ & + 7023914883t^{10} + 3057882354t^{11} + 1093133274t^{12} \\ & + 320002575t^{13} + 76243422t^{14} + 14633941t^{15} + 2227253t^{16} \\ & + 262453t^{17} + 23075t^{18} + 1424t^{19} + 55t^{20} + t^{21} \,. \end{aligned}$$

Considered as an equation over $A = R[\![t]\!]$, this describes the 7-cyclic Galois covering[18]

$$\phi : \mathsf{Y} \to \mathsf{X}$$

of open analytic curves, with the open unit disk X being the residue class of $\bar{x}_1$, with $\mathsf{Y} := \Phi^{\mathrm{rig}^{-1}}(\mathsf{X})$, and with $\phi := \Phi^{\mathrm{rig}}|_{\mathsf{Y}}$. The covering ϕ is ramified over three points (corresponding to the zeros of $f(t)$ with positive valuation). We are thus in the situation of Sect. 4.1. As explained there, the first step is to construct the minimal exhausting disk containing all ramification points. For this, we have to study the étale covering of the open annulus $\mathsf{A} := \mathsf{X} \setminus \mathsf{D}_{\mathrm{ram}}$, with $\mathsf{D}_{\mathrm{ram}}$ denoting the smallest closed subdisk of X containing all ramification points.

Since the ramification points are of valuation $1/3$ and their pairwise distance is of valuation $1/2$, we have

$$\mathsf{D}_{\mathrm{ram}} := \{x \in \mathsf{X} \mid 1/3 \le v(t(x))\}\,;$$

in particular, we can take t as a parameter for A, that is,

$$\mathsf{A} = \{x \in \mathsf{X} \mid 0 < v(t(x)) < 1/3\}\,.$$

The corresponding ring of zero-bounded analytic functions is $A := R[\![s, t \mid st = a]\!]$, where $a \in K$ is an element of valuation $v(a) = 1/3$; we take a to be a third root of 7 and hence assume K to contain the field

$$K_1 := \mathbb{Q}_7(a \mid a^3 = 7)\,.$$

With respect to the parameter t, the coefficients of $f = \sum_{i=0}^{21} a_i t^i$ have respective valuation

$$[1,2,1,0,0,0,0,0,1,1,0,0,0,0,0,1,1,0,0,0,0,0]\,.$$

In the notation of Sect. 4.2.1, we thus have $i_0 = 3 \neq 0$. As a consequence, A is immediately recognized to be separating, and a first improving blow-up of $\mathsf{Y} \to \mathsf{X}$ corresponds to the affinoid disk $\mathsf{D}_{\mathrm{ram}}$. To get a description of the corresponding exceptional divisor, we first express f in terms of

$$t_1 := \frac{t}{a}$$

[18]Here, the analog of Footnote 8 also applies: ϕ is only a Galois covering when K contains a primitive seventh root of unity; for our techniques to be applicable, it suffices to have the covering given by an equation of Kummer type.

(which serves as a parameter for the closed disk $\mathsf{D}_{\mathrm{ram}}$) and then normalize the polynomial $f(t_1) = 33787663 + \dots$ by dividing through the zeroth coefficient $a_0 = 33787663$ to obtain

$$\begin{aligned} f_1 := 1 + 140a/13\, t_1 + 9172a^2/169\, t_1^2 + 1618475/2197\, t_1^3 \\ + \text{ terms in } t_1 \text{ of positive valuation} . \end{aligned} \tag{4.17}$$

Assuming K to contain

$$K_2 := K_1(\lambda \mid \lambda^7 = a_0) \,,$$

the substitution $y_2 := y_1/\lambda$ leads to the equation

$$y_2^7 = f_1(t_1) \,, \tag{4.18}$$

which reduces to the irreducible and purely inseparable equation

$$\bar{y}_2^7 = 1 + \bar{t}_1^3$$

over $k[\bar{t}_1]$. As this describes the exceptional divisor of the modification induced by $\mathsf{D}_{\mathrm{ram}}$, the only remaining critical point is at $\bar{t}_1 = 0$ (corresponding to the only zero of the differential $\mathrm{d}(1 + \bar{t}_1^3)$).

4.6.3. *Étale Covering of an Open Subdisk.* Accordingly, we have to examine the residue class of $\bar{t}_1 = 0$—that is, the open disk

$$\mathsf{X}_1 := \{x \in \mathsf{X} \mid 0 < v(t_1(x))\} \subset \mathsf{X}$$

—and the induced covering $\phi_1 : \mathsf{Y}_1 := \phi^{-1}(\mathsf{X}_1) \to \mathsf{X}_1$ thereof, which is described by Eq. (4.18) considered as an equation over the power series ring $R[\![t_1]\!] = \mathring{\mathscr{O}}_{\mathsf{X}_1}$.

It is immediate from (4.17) that 1 is a sufficiently precise approximation of $f_1(t_1)$. As the first four coefficients of $f_1 - 1^7$ have respective valuation

$$\left[\infty, \frac{4}{3}, \frac{2}{3}, 0\right] \,,$$

the critical line segment of f_1 is given by $\overline{P_0' P_3}$ (note that the first coefficient $140a/13$ is small enough to not interfere with this segment). Consequently, our resolution algorithm will terminate with a smooth component of positive genus. More precisely, the critical radius is $\rho_1 = 7/18$, so a parameter for the critical disk $\mathsf{D}_1 \subset \mathsf{X}_1$ is given by $t_2 := t_1/ab$, where $b \in K$ is any element of valuation $v(b) = 1/18$; we assume K to contain

$$K_3 := K_2(b \mid b^6 = a) \,.$$

As b^3 is an element of valuation $v(b^3) = 1/6$, the usual variable change

$$w := \frac{y_2 - 1}{b^3} \,,$$

leads to an equation that reduces to the irreducible Artin–Schreier equation

$$\bar{w}^7 + \bar{w} = \bar{t}_2^3 \tag{4.19}$$

over $k[\bar{t}_2]$. We have thus found a smooth component Y_1 of genus three (lying over a rational component X_1).

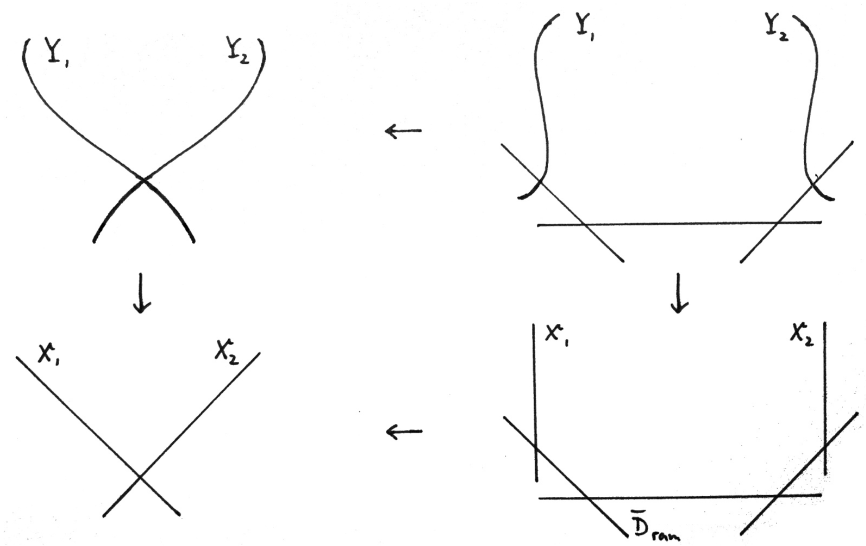

FIGURE 4.4. The semistable models of Y and X, as produced by our algorithm, are depicted on the right hand side. The stable model for Y, resulting from blowing-down superfluous rational components, is shown on the left (together with the corresponding quotient model for X).

4.6.4. *Stable Model and Monodromy Action.* All in all, we have obtained the following result (illustrated in Fig. 4.4).

Claim 4.22. *With notation as above, the totally ramified field extension $K_3/\mathbb{Q}_7$ of degree $[K_3 : \mathbb{Q}_7] = 126$ realizes the stable model Y^{ss} of Y. The stable reduction is given by two smooth components $Y_1, Y_2 \subset \bar{Y}$ of genus three intersecting in a single ordinary double point.* □

Remark 4.23. One reason why it was so easy to determine the stable model of Y is that the invariant $m = 6$ (corresponding to the ramification of the covering) is strictly smaller than the characteristic $p = 7$ of the residue field. For p-cyclic coverings with $m < p$—and under the assumption of equidistant geometry—Lehr [Leh01] has already given an explicit construction of corresponding semistable models. We therefore expected the above situation to be more simple than the general case (even without having equidistant geometry).

We want to show that $K_3/\mathbb{Q}_7$ is the minimal extension needed for realizing the stable reduction of Y (in the sense of Def. 1.4). For this, we study the compositum $M := K_3\mathbb{Q}_7^{\mathrm{nr}}$ with the maximal unramified extension $\mathbb{Q}_7^{\mathrm{nr}}/\mathbb{Q}_7$. Since $\mathbb{Q}_7^{\mathrm{nr}}$ contains the eighteenth roots of unity, the subfield

$$L := \mathbb{Q}_7^{\mathrm{nr}}(b \mid b^{18} = 7) \subset M$$

is a cyclic Galois extension of degree eighteen. It is uniquely determined because the ramification is tame. In particular, L is seen to contain a primitive seventh root ξ of unity: the extension degree $[L : \mathbb{Q}_7^{\mathrm{nr}}] = 18$ is divisible by 6, which is the degree of ξ over $\mathbb{Q}_7^{\mathrm{nr}}$. Consequently, the wildly ramified extension of degree seven,

$$M = L(\lambda \mid \lambda^7 = a_0)\,,$$

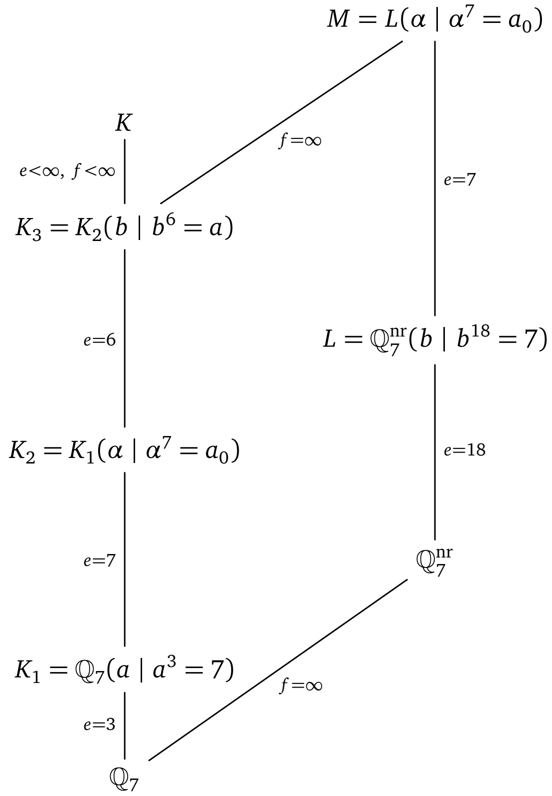

FIGURE 4.5. The tower of fields built up in the course of applying our algorithm is shown on the left hand side; the extensions of the maximal unramified extension used for the study of the monodromy action are shown on the right hand side. $M/\mathbb{Q}_7^{\mathrm{nr}}$ is a totally ramified Galois extension of degree $[M : \mathbb{Q}_7^{\mathrm{nr}}] = 126$; the subfield $L \subset M$ corresponds to the maximal tamely ramified subextension of degree 18 over $\mathbb{Q}_7^{\mathrm{nr}}$.

with $a_0 = 33787663 \in \mathbb{Q}_7$, is also cyclic. As M is the splitting field of the separable polynomials $t^{18} - 7$ and $t^7 - a_0$ over $\mathbb{Q}_7^{\mathrm{nr}}$, we have that $M/\mathbb{Q}_7^{\mathrm{nr}}$ is Galois of degree 126. We write $\Gamma := \mathrm{Gal}(M/\mathbb{Q}_7^{\mathrm{nr}})$ for the Galois group of the total extension, $T := \mathrm{Gal}(M/L)$ for the wild ramification subgroup, and $P := \mathrm{Gal}(L/\mathbb{Q}_7^{\mathrm{nr}})$ for the cyclic quotient corresponding to the maximal tamely ramified subextension. A diagram of the involved field extensions can be found in Fig. 4.5.

In the following, we will consider Y and X as curves over $\mathbb{Q}_7^{\mathrm{nr}}$, and Y^{ss} as a model for $Y \otimes_{\mathbb{Q}_7^{\mathrm{nr}}} M$. As usual, we write X^{ss} for the corresponding quotient model of X.

Claim 4.24. *With notation as above, the Galois extension $M/\mathbb{Q}_7^{\mathrm{nr}}$ of degree 126 is the monodromy extension of Y.*

Proof. We have to show that Γ acts faithfully on the special fiber of Y^{ss}. As the centers of the modifications leading to the separable genus-3-components $Y_1, Y_2 \in \bar{Y}$ are defined over $\mathbb{Q}_7^{\mathrm{nr}}$ (namely, $x_1 = 3$ and $x_2 = 5$), the action of Γ does not permute the two components and therefore restricts to an action on each of these; we will see that already the restricted actions are faithful.

The Artin–Schreier type equation (4.19) describes the component $Y_1 \subset \bar{Y}$ that we have found with our algorithm; Y_1 lies above the rational component $X_1 \subset \bar{X}$, which is determined by the parameter $t_2 = t/b^7$. As t_2 is already defined over the subfield $L \subset M$, the induced action of Γ on X_1 is via the cyclic quotient $P = \mathrm{Gal}(L/\mathbb{Q}_7^{\mathrm{nr}})$. Choose some generator $\sigma \in P$; then $\sigma(b) = \zeta b$, with $\zeta \in L$ a primitive eighteenth root of unity. We calculate

$$\sigma(t_2) = \frac{t}{\sigma(b^7)} = \frac{t}{\zeta^7 b^7} = t_2 \zeta^{11}\,.$$

As $\operatorname{ord}\zeta = 18$ is prime to the residue field characteristic $p = 7$, the reduction of ζ remains an element of order eighteen, and so does the reduction of ζ^{11}. We deduce that P acts faithfully on X_1. Consequently, the action of Γ on X_1 has kernel precisely $T = \mathrm{Gal}(M/L)$.

It remains to show that T acts faithfully on Y_1, which is defined in terms of the variable $w = (y_2 - 1)/b^3$, with $y_2 = y_1/\lambda$. Denote a generator of the 7-cyclic group $T = \mathrm{Gal}(M/L)$ by τ; it sends $\lambda \in M$ to $\xi\lambda$, with ξ a primitive seventh root of unity, and fixes b. We calculate

$$\tau(y_2) = \frac{y_1}{\tau(\lambda)} = \frac{y_1}{\xi\lambda} = \frac{y_2}{\xi}$$

and

$$\tau(w) = \frac{\tau(y_2) - 1}{b^3} = \frac{y_2/\xi - 1}{b^3} = \frac{(b^3 w + 1)/\xi - 1}{b^3} = \frac{w}{\xi} + \frac{1-\xi}{b^3\xi}\,.$$

Note that $v(1-\xi) = 1/6 = v(b^3)$, so $(1-\xi)/b^3\xi$ reduces to a non-zero element u of $\mathbb{F}_7^{\mathrm{ac}}$ (the residue field of M). As the seventh root of unity ξ reduces to 1, we have

$$\tau(\bar{w}) = \bar{w} + u\,,$$

and τ is seen to have order seven on Y_1; that is, the action of T on Y_1 is faithful.

Taking the above results together, we obtain that $\Gamma = \mathrm{Gal}(M/\mathbb{Q}_7^{\mathrm{nr}})$ acts faithfully on Y_1 and, a fortiori, on the stable model of Y^{ss}. Hence, $M/\mathbb{Q}_7^{\mathrm{nr}}$ is the minimal field extension realizing the stable reduction of Y (considered as a curve over $\mathbb{Q}_7^{\mathrm{nr}}$). □

5. Cyclic Galois Covers of Prime Power Degree

In Sect. 2, we were able to construct the minimal exhausting disk for prime-cyclic étale coverings of the open unit disk; together with the results from Sect. 4 on coverings of open annuli, this allows us to also deal with a cyclic covering of prime power degree by splitting the covering into prime-cyclic substeps (which can then be treated with the methods already described).

5.1. **Reduction to Prime-Cyclic Covers.** As in Sects. 2 and 4, we will assume K to be of mixed characteristic $(0, p)$. Let $n \in \mathbb{N}$ with $n \geq 2$, and consider an étale p^n-cyclic Galois cover $\phi : \mathsf{Y} \to \mathsf{X}$ of the open unit disk. As usual, we denote by $A := \mathring{\mathscr{O}}_{\mathsf{X}}$ the ring of zero-bounded analytic functions on the open disk X; choosing a disk parameter t, we can write $A = R[\![t]\!]$. Assuming K to contain a primitive p^nth root of unity, we can further assume that the covering is given by a Kummer equation

$$y^{p^n} = f\,,$$

with $f \in A^\times$; cf. the reasoning in Sect. 2.1. We introduce new variables $z_1, \ldots, z_{n-1}$ and consider the system of equations

$$z_1^p = f, \quad z_2^p = z_1, \quad \ldots\,, \quad y^p = z_{n-1}\,;$$

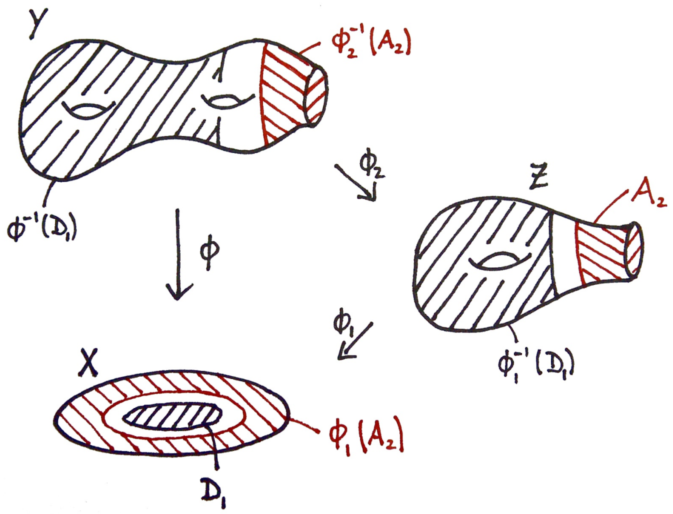

FIGURE 5.1. A p^2-cyclic covering $\phi : \mathsf{Y} \to \mathsf{X}$ of the open unit disk can be handled by splitting the covering into two successive p-cyclic substeps $\phi_2 : \mathsf{Y} \to \mathsf{Z}$ and $\phi_1 : \mathsf{Z} \to \mathsf{X}$. The minimal exhausting disk D_1 with respect to ϕ_1 might be too small with respect to the total covering ϕ; it becomes necessary to examine the covering of the open annulus $\phi_1^{-1}(\mathsf{X} \setminus \mathsf{D}_1)$ and to determine the corresponding maximal separating boundary domain A_2 with respect to ϕ_2.

this splits the covering ϕ into n subcoverings of open analytic curves

$$\phi_1 : \mathsf{Z}_1 \to \mathsf{X}, \quad \dots, \quad \phi_n : \mathsf{Y} \to \mathsf{Z}_{n-1},$$

each cyclic of order p. We will handle these subcoverings one after another, starting with ϕ_1; note that $Z_1, \dots, Z_{n-1}$ will usually neither be open disks nor open annuli.

The first step is to consider the covering $\phi_1 : \mathsf{Z}_1 \to \mathsf{X}$ given by $z_1^p = f$. More precisely, the open analytic curve Z_1 corresponds to the integral closure of A in the field extension of Frac$A$ given by $z_1^p = f$. The covering ϕ_1 is a p-cyclic étale covering of the open unit disk and can thus be handled with the methods from Sect. 2. In particular, we can find the minimal exhausting disk $\mathsf{D}_1 \subset \mathsf{X}$ with respect to ϕ_1. With regard to the composed covering $\phi_2 \circ \phi_1 : \mathsf{Z}_2 \to \mathsf{X}$, however, the affinoid disk D_1 might be too small to be exhausting. In other words, although the preimage

$$\mathsf{V}_1 := \phi_1^{-1}(\mathsf{A}_1^{[1]}) \subset \mathsf{Z}_1$$

of the ϕ_1-separating boundary domain $\mathsf{A}_1^{[1]} := \mathsf{X} \setminus \mathsf{D}_1$ is a single open annulus, the latter does not have to be separating with respect to $\phi_2 : \mathsf{Z}_2 \to \mathsf{Z}_1$; this situation is depicted in Fig. 5.1. It hence becomes necessary to study the induced covering of the open annulus V_1 in detail and to determine the maximal separating boundary domain $\mathsf{A}_2 \subset \mathsf{V}_1$ (with respect to the boundary point of V_1 lying above the unique boundary point of the open disk X). By the results of [AW11, Sect. 4.6],

$$\mathsf{A}_1^{[2]} := \phi_1(\mathsf{A}_2) \subset \mathsf{A}_1^{[1]} \subset \mathsf{X}$$

will be a separating boundary domain with respect to $\phi_2 \circ \phi_1$ and it will also be maximal with this property.

The remaining steps (if there are any left) are similar. Since A_2 is separating for the étale covering of the open annulus V_1, the preimage $\mathsf{V}_2 := \phi_2^{-1}(\mathsf{A}_2)$ is a disjoint union of (isomorphic) open annuli. We pick one of them and determine the maximal separating boundary domain with respect to the restricted ϕ_3; this also determines the maximal separating boundary domain for the composed cover $\phi_3 \circ \phi_2 \circ \phi_1$. Continuing this way, we will finally end up with a boundary domain $\mathsf{A}_1^{[n]} \subset \mathsf{X}$ that is maximal separating with respect to ϕ. The modification induced by the corresponding minimal exhausting disk will then improve the model of Y.

For doing actual computations, it will be necessary to write the equations for the subcoverings in terms of suitable parameters. For example, with notation as above, the second step requires to get hold of a parameter for the open annulus $\mathsf{V}_1 \subset \mathsf{Z}_1$ and to write the equation corresponding to the covering ϕ_2 (restricted to $\phi_2^{-1}(\mathsf{V}_1)$) in terms of this parameter.

Remark 5.1. The above situation will generally require to handle true power series and Laurent series. This is in contrast to those situations that arise by localizing a global p-cyclic covering of $\mathbb{P}^1_K$ in critical points (where we can get by with polynomial equations).

Remark 5.2. The way to handle the ramified situation is similar: in this case, one has to introduce intermediate steps by first constructing the minimal exhausting disk containing all ramification points; cf. the p-cyclic situation in Sect. 4.1. More generally, the approach of Sect. 5.1 allows to determine the stable reduction of an arbitrary cyclic covering $Y \to X$ of prime power degree, provided one has a semistable model for X: one reduces to the local situation and studies appropriate prime-cyclic substeps, which will be coverings of open disks or open annuli (and which can therefore be handled with the methods described in Sects. 2 and 4 of the present paper).

5.2. **Example: Cyclic Galois Cover of Prime Square Degree.** To exemplify the approach described in Sect. 5.1, we will thoroughly study a p^2-cyclic covering of the open unit disk. Though the chosen example is quite simple, it illustrates the key elements of our method and should be well-suited to convey the main ideas.

5.2.1. *Setting.* Let $\Phi : Y \to X := \mathbb{P}^1_{\mathbb{Q}_3}$ be the 3^2-cyclic covering of the projective line that is given by the Kummer type equation

$$y^9 = 1 + t^2 ; \tag{5.1}$$

by the Riemann–Hurwitz formula [Har77, Cor. IV.2.4], the curve Y is seen to be of genus $g_Y = 4$. As in all examples treated before, we will study the covering over a sufficiently large finite extension K of $\mathbb{Q}_3$, with corresponding valuation ring R and residue field k; a posteriori, we will be able to specify a field extension that works for us.

We start our resolution algorithm by taking the standard smooth model X_R for $\mathbb{P}^1_K$. As the right hand side of Eq. (5.1) is not a third power in reduction, the equation describes the normalization Y_R of X_R in Y (cf. the reasoning using Serre's criterion in the proof of Prop. 2.2). The corresponding special fiber $\bar{Y}$ consists of the irreducible curve with equation $\bar{y}^9 = 1 + \bar{t}^2$. Since $\bar{t} = 0$ is the unique zero of the differential $\mathrm{d}(1 + \bar{t}^2) = 2\bar{t}\mathrm{d}\bar{t}$, this curve has precisely one singular point; namely, the point with affine coordinates $(\bar{t}, \bar{y}) = (0,0)$. We can hence restrict ourselves to the study of the 9-cyclic étale Galois covering

$$\phi : \mathsf{Y} \to \mathsf{X} ,$$

where the open unit disk $\mathsf{X} :=]0[_{X_R}$ is the residue class of the critical point $0 \in k$, where Y is the preimage of X under Φ^{rig}, and where the covering is given by Eq. (5.1) considered as an equation over $A = R[\![t]\!] = \mathring{\mathcal{O}}_{\mathsf{X}}$. Let

$$\phi_1 : \mathsf{Z} \to \mathsf{X} \quad \text{with } z^3 = 1 + t^2 \tag{5.2}$$

and

$$\phi_2 : \mathsf{Y} \to \mathsf{Z} \quad \text{with } y^3 = z \tag{5.3}$$

be the p-cyclic substeps. Note that Z is not an open disk but an open analytic curve of genus one.

5.2.2. *Minimal Exhausting Disk.* Obviously, the minimal exhausting disk with respect to the 3-cyclic étale covering ϕ_1 is

$$\mathsf{D}_1 := \{x \in \mathsf{X} \mid 3/4 \leq v(t(x))\} ,$$

as $h = 1$ is seen to give a sufficiently precise approximation of the polynomial $1 + t^2$ (which describes the covering ϕ_1 via (5.2)); note that $t = 0$ serves as a well-suited center, as the critical segment $\overline{P_0' P_2}$ of the approximation is not determined by P_1. Denote the corresponding maximal separating boundary domain by $\mathsf{A}_1 := \mathsf{X} \backslash \mathsf{D}_1$; this is an open annulus of thickness $3/4$. The disk parameter t also serves as a parameter for A_1 when considered as an element of the corresponding ring of zero-bounded analytic functions

$$A_1 := \mathring{\mathcal{O}}_{\mathsf{A}_1} = R[\![s, t \mid st = a]\!] ,$$

where $a \in K$ is an element of valuation $3/4$. Consequently, we need K to contain elements of valuation $1/4$; we assume K to contain a fourth root of 3—that is, to contain the field

$$K_1 := \mathbb{Q}_3(\alpha \mid \alpha^4 = 3) .$$

We can then take $a := \alpha^3$.

The preimage $\mathsf{V}_1 := \phi_1^{-1}(\mathsf{A}_1)$ of the maximal ϕ_1-separating boundary domain is a single open annulus of thickness $1/4$ and described by the ring $B_1 = A_1[w]$, with

$$w := z - 1 \tag{5.4}$$

satisfying the irreducible equation

$$w^3 + 3w^2 + 3w = t^2 \tag{5.5}$$

over A_1; see the proof of Lem. 2.34. By Lem. 1.31 and its proof, the element

$$v := \frac{t}{w} \in B_1 \tag{5.6}$$

(which is as in (1.2)) can be used as a parameter for the open annulus V_1. To determine the maximal separating boundary domain with respect to ϕ_2 (restricted to the ϕ_2-preimage of V_1), the corresponding equation (5.3) has to be rewritten in terms of v. This will require us to express t as a Laurent series in v. Substituting $w = t/v$ into Eq. (5.5) yields the relation

$$t^2 v^{-3} + 3tv^{-2} + 3v^{-1} = t . \tag{5.7}$$

It is clear that iterative substitution of Eq. (5.7) into itself leads to a power series expansion of t in terms of v^{-1}:

$$\begin{aligned} t &= 3v^{-1} + 3tv^{-2} + t^2v^{-3} \\ &= 3v^{-1} + 3(3v^{-1} + 3tv^{-2} + t^2v^{-3})v^{-2} + (3v^{-1} + 3tv^{-2} + t^2v^{-3})^2v^{-3} \\ &= \cdots = 3v^{-1} + 9v^{-3} + 36v^{-5} + 162v^{-7} + 783v^{-9} + 3969v^{-11} + \ldots . \end{aligned} \tag{5.8}$$

Plugging the relations (5.4), (5.6), and (5.8) into Eq. (5.2), we obtain

$$y^3 = z = w + 1 = tv^{-1} + 1 \tag{5.9}$$

$$= \cdots + 3969v^{-12} + 783v^{-10} + 162v^{-8} + 36v^{-6} + 9v^{-4} + 3v^{-2} + 1\,. \tag{5.10}$$

The valuations

$$[\ldots, 4, \infty, 3, \infty, 4, \infty, 2, \infty, 2, \infty, 1, \infty, 0]$$

of the coefficients of $v^{-12}, \ldots, v^0$ suggest that $h = 1$ gives a sufficiently precise approximation of the right hand side of (5.10), with critical point P_{-2} and a critical segment of slope $-1/4$ (resulting in a critical thickness $1/4$). Indeed, by (5.9), our assertion on the critical thickness is equivalent to showing that $v_{1/4}(w) = 1/2$ (where the valuation $v_{1/4}$ is meant with respect to the parameter v of the open annulus V_1, that is, $v_{1/4}(v) = 1/4$). Substituting $t = vw$ from Eq. (5.6) into Eq. (5.5) and then dividing by w^2, we obtain

$$w + 3 + 3w^{-1} = v^2\,.$$

Using the strong triangle inequality, it immediately follows that both $v_{1/4}(w) < 1/2$ and $v_{1/4}(w) > 1/2$ are not possible; hence, $v_{1/4}(w) = 1/2$ as asserted.

The critical thickness being $1/4$ means that we are in the trivial case (1) of Def. 4.5; in other words, V_1 as a whole is recognized to be separating, with exactly one open annulus lying above. Consequently, A_1 is recognized as the maximal separating boundary domain with respect to the total covering $\phi = \phi_2 \circ \phi_1$. See Fig. 5.2 for a sketch of the situation.

5.2.3. *Describing the Modification.* Now that we have found the minimal ϕ-exhausting disk, we want to describe the effect of the corresponding modification and determine the stable model of Y.

The element $t_1 := t/\alpha^3$ can be taken as a parameter for the affinoid disk D_1. The preimage $\mathsf{E}_1 := \phi^{-1}(\mathsf{D}_1) \subset \mathsf{Z}$ is an affinoid with good reduction, given by the equation

$$z_1^3 + \alpha^2 z_1^2 + z_1 = t_1^2 \tag{5.11}$$

obtained from Eq. (5.2) by substituting

$$z_1 := \frac{z-1}{\alpha^2}\,. \tag{5.12}$$

The corresponding special fiber is a smooth curve of genus one, given by the Artin–Schreier type equation

$$\bar{z}_1^3 + \bar{z}_1 = \bar{t}_1^2\,. \tag{5.13}$$

By Eqs. (5.3) and (5.12), the covering ϕ_2 (restricted to $\mathsf{E}_2 := \phi_2^{-1}(\mathsf{E}_1)$) is given by

$$y^3 = z = 1 + \alpha^2 z_1\,. \tag{5.14}$$

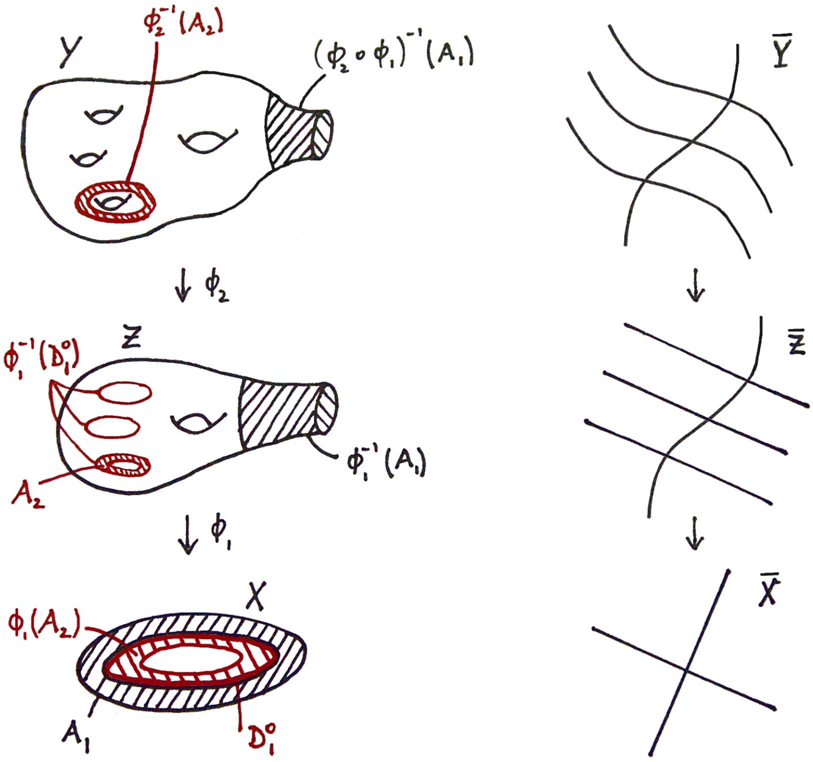

FIGURE 5.2. The 9-cyclic covering $\phi : \mathsf{Y} \to \mathsf{X}$ of the rigid analytic open unit disk is split into two 3-cyclic substeps $\phi_1 : \mathsf{Z} \to \mathsf{X}$ and $\phi_2 : \mathsf{Y} \to \mathsf{Z}$. The minimal ϕ_1-exhausting disk D_1 is also exhausting with respect to ϕ, as a single open annulus is lying above $\phi_1^{-1}(\mathsf{A}_1)$. The preimage in Z of the interior D_1° is given by three isomorphic copies, and it suffices to determine the maximal separating boundary domain A_2 for one of them. In the end, we obtain that the stable reduction $\bar{Y}$ of Y is given by four components of genus one, with three of them each intersecting the fourth in an ordinary double point.

Note that z_1 is by Eq. (5.13) a unit with respect to the inf-valuation on E_2. Assuming K to contain an element $\lambda \in K$ of valuation $v(\lambda) = 1/6$, the usual substitution

$$y_1 := \frac{y-1}{\lambda}$$

then transforms Eq. (5.14) into an equation having irreducible reduction

$$\bar{y}_1^3 = \bar{z}_1 \,, \tag{5.15}$$

which therefore describes the special fiber of E_2 (cf. the argument in the proof of Prop. 2.2 using Serre's criterion on normality). As noted, K must contain an element of valuation $1/6$; since K already contains elements of valuation $1/4$, this leads to a further totally ramified extension of degree three. We will suppose K to contain the field

$$K_2 := K_1(\beta \mid \beta^3 = \alpha) \,;$$

then $\lambda = \beta^2$ is an element of valuation $1/6$, as needed.

The remaining critical points with respect to ϕ correspond to the zeros of the differential of the right hand side of Eq. (5.15); using the relation from (5.13) between $\bar{z}_1$ and $\bar{t}_1$, we calculate

$$\mathrm{d}\bar{z}_1 = \mathrm{d}(\bar{t}_1^2) = 2\bar{t}_1\mathrm{d}\bar{t}_1 \,. \tag{5.16}$$

It follows that $\bar{t}_1 = 0$ is the only critical point on the reduction of D_1. To describe the critical points on the reduction of E_1, the residue field k has to contain an element i with $i^2 = -1$; that is, the base field K is required to contain the (unique) unramified extension

$$\mathbb{Q}_9 := \mathbb{Q}_3(i \mid i^2 = -1)\,,$$

of degree two. There are then exactly three critical points on the reduction of E_1; namely, by Eq. (5.13), the points with coordinates $(\bar{t}_1, \bar{z}_1) = (0,0)$, $(0,i)$, and $(0,-i)$. In other words, the ϕ_1-preimage of the critical disk

$$\mathsf{D}_1^\circ := \{x \in \mathsf{D}_1 \mid 0 < v(t_1(x))\} = \{x \in \mathsf{X} \mid 3/4 < v(t(x))\}$$

corresponding to $\bar{t}_1 = 0$ is the disjoint union of three copies of D_1°. It hence suffices to study the covering ϕ_2 restricted to the preimage of one of these disks. Already now, we can deduce that the next improving modification will give a smooth elliptic component because the differential (5.16) has only a single zero of order one. We thus have established the following result.

Claim 5.3. *The stable reduction $\bar{Y}$ of Y consists of four elliptic components, with three of them each intersecting the fourth in an ordinary double point; see the illustration in Fig. 5.2.* □

Remark 5.4. As we have found all four components of the stable reduction $\bar{Y}$ by considering the ϕ-preimage of D_1 and the covering of D_1°, we could have skipped the calculations from Sect. 5.2.2. Namely, it follows already from genus considerations that a model for Y with four elliptic components must be semistable; in particular, the boundary domain $\mathsf{A}_1 = \mathsf{X} \setminus \mathsf{D}_1$ has to be ϕ-separating.

5.2.4. *Final Modification.* To make the final blow-up explicit, we let Z_1 denote one of the open disks in the ϕ_1-preimage of D_1°, say the residue class in E_1 of the point $(\bar{t}_1, \bar{z}_1) = (0,0)$. The covering ϕ_2 (restricted to the preimage of Z_1) is given by Eq. (5.14), now considered as an equation over the ring of zero-bounded analytic functions on Z_1. We need to write the right hand side of this equation as a power series in t_1 (which serves as a parameter both for D_1° and the isomorphic Z_1). For this, the explicit relation between z_1 and t_1, given by Eq. (5.11), has to be used. Analog to the situation in Sect. 5.2.2, it is clear that iterated substitution of Eq. (5.11) in itself leads to a power series expansion of z_1 in terms of t_1:

$$\begin{aligned} z_1 &= t_1^2 - 3z_1^2 - z_1^3 \\ &= t_1^2 - 3(t_1^2 - 3z_1^2 - z_1^3)^2 - (t_1^2 - 3z_1^2 - z_1^3)^3 \\ &= t_1^2 - 3t_1^4 + \dots \,. \end{aligned}$$

Only the first term t_1^2 is relevant for our calculations, as the critical line segment for the resulting equation

$$y^3 = 1 + \alpha^2 z_1 = 1 + \alpha^2(t_1^2 - 3t_1^4 + \dots)$$

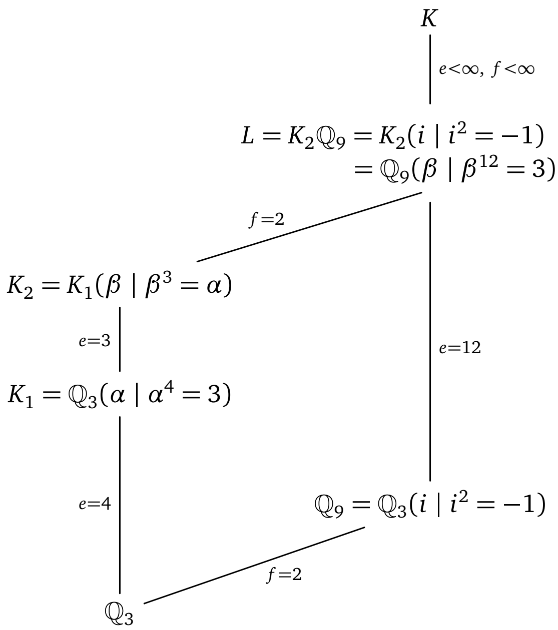

FIGURE 5.3. The tower of fields built up in the course of applying our algorithm. The ramified part $L/\mathbb{Q}_9$ is of degree twelve.

is immediately seen to be $\overline{P_0' P_2} = \overline{(0,3/2)(2,1/2)}$. This gives a critical radius $1/2$ with respect to t_1 (or $3/4+1/2=5/4$ with respect to t) and leads to a smooth genus-one-component in reduction. Description of the latter component requires an element of valuation $1/2$; such elements are already contained in K_2.

Our algorithm has thus produced the extension field $L := K_2\mathbb{Q}_9$ of $\mathbb{Q}_3$ over which a semistable model of Y can be found; see the field diagram in Fig. 5.3. The extension is of degree $[L : \mathbb{Q}_3] = 24$ (with $e = 12$ and $f = 2$); with respect to the Semistable Reduction Theorem, only the ramified part is of interest and we obtain:

Claim 5.5. *The stable model of Y can be realized over $L = \mathbb{Q}_9(\beta \mid \beta^{12} = 3)$, which is a totally ramified extension of $\mathbb{Q}_9$ of degree twelve.*

Remark 5.6. The above example was quite simple as no serious calculations were involved; in particular, it was not necessary to do complicated p-Taylor approximations. Nevertheless, all steps for determining the semistable model of a cyclic covering with prime power degree get fairly well illustrated, and it should be clear how to proceed in more complicated situations.

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